

AD

780 811

TECHNICAL REPORT

74-26-PR

# RESPONSE OF POLYMERS TO TENSILE IMPACT

2. EXTENSION of the INTEGRAL-EQUATION,  
SUCCESSIVE-SUBSTITUTION SOLUTION to NON-LINEAR,  
TIME-INDEPENDENT MATERIALS

by

Prescott D. Crout

Massachusetts Institute of Technology

and

Malcolm N. Pilsworth, Jr.

Harold J. Hoge

US Army Natick Laboratories

February 1974

UNITED STATES ARMY  
NATICK LABORATORIES  
Natick, Massachusetts 01760



Pioneering Research Laboratory

Approved for public release; distribution unlimited.

Citation of trade names in this report does not constitute an official indorsement or approval of the use of such items.

Destroy this report when no longer needed. Do not return it to the originator.

This document has been approved  
for public release and sale; its  
distribution is unlimited.

AD \_\_\_\_\_

TECHNICAL REPORT

74-26-PR

**Response of Polymers to Tensile Impact.  
2. Extension of the Integral-Equation,  
Successive-Substitution Solution to Non-  
Linear, Time-Independent Materials**

by

Prescott D. Crout

Massachusetts Institute of Technology

and

Malcolm N. Pilsworth, Jr.

Harold J. Hoge

U. S. Army Natick Laboratories

Project Reference:  
1T061102B11A-07

February 1974

Pioneering Research Laboratory  
US ARMY NATICK LABORATORIES  
Natick, Massachusetts 01760

## FOREWORD

This is the second in a series of reports on high-speed tensile-impact work performed in the Pioneering Research Laboratory of NLABS. The first report (NLABS Tech. Rept 69-18-PR) appeared over 5 years ago. It had been planned to publish all three of the projected reports at that time, but for various reasons the second and third reports were delayed.

The first report dealt with the basic mathematical techniques and their application to a linear material. The present report deals with non-linear materials which do not exhibit creep or other time dependence. The third report is planned to deal with non-linear materials that do exhibit creep. Since all actual materials exhibit creep it is only in the third report that we can hope to demonstrate good agreement between theory and experiment.

Body armor must be resistant to high-speed impact in order to stop projectiles and shell fragments. Parachute straps are subjected to sudden tensile strains when the parachute opens and the falling load is suddenly decelerated. The mathematical techniques described in the present paper should be useful in finding out more about how materials fail under rapid impact. With this knowledge, better materials can almost certainly be developed.

## CONTENTS

	<u>Page</u>
List of Figures	iv
List of Tables	v
Nomenclature	vi
Abstract	viii
1. Introduction	1
2. Summary of Report I	1
3. Extension of the Solution to the Non-Linear Case	7
4. Description of the Repetitive Process when Applied to the Non-Linear String	9
5. Results of Digital Computer Calcula- tions for a Non-Linear Model	14
6. Nature of the Oscillations when the Stress-Strain Curve is Concave Upward	23
7. Theoretical Investigations Pertaining to Oscillations, Momentum Theorem	24
8. Conclusion	61
9. Appendix 1	62
10. Appendix 2	71
11. References	76

## LIST OF FIGURES

<u>Fig.</u>		<u>Page</u>
1.	String subjected to tensile impact.	3
2.	Strain for a linear time-independent model as obtained from digital computer calculations for $n = 60$ with $j = 105$ ( $t = 1.75$ ). The velocity of the impacted end increased linearly to $v_0$ at $t = 0.5$ .	6
3.	Lattice of points at which calculations are to be made.	11
4.	Dynamic stress-strain curve for nylon yarn.	15
5.	Displacement and velocity of impacted end of string.	16
6.	Strain pattern in nylon yarn calculated by successive substitution for a time-independent model. Time interval between lines is 0.0417 millisec.	18
7.	Strain pattern for nylon yarn calculated by method-of-characteristics.	19
8.	Strain pattern for nylon yarn calculated by successive substitution (lines) and by method-of-characteristics (points).	21
9.	Closed surfaces for momentum theorem. At time $t$ , $S$ contains a given mass. At $t + dt$ mass has moved to $S_2$ and position of $S$ has moved to $S_1$ .	25
10.	Strain wave with vertical front.	28
11.	One vertical wave front being overtaken by another.	28
12.	Stress-strain curve showing chords that determine velocities of vertical wave fronts.	34
13.	Wave front of arbitrary but constant shape.	34
14.	Vertical wave front being overtaken by an infinitesimal wave.	39

<u>Fig.</u>		<u>Page</u>
15.	String composed of discrete masses connected by slack cords.	42
16.	Stress-strain curve of string in Fig. 15.	42
17.	Displacement-time curve for string of Fig. 15 subjected to constant-velocity impact.	43
18.	Displacement-time curve for string of Fig. 15 subjected to an impact that changes in velocity.	44
19.	String composed of discrete masses and linear springs.	46
20.	Wave-front as a transition region between the unstretched and stretched string.	56
21.	Strain-wave front composed of incremental strains each moving at its own velocity.	64
22.	Strain-wave front with negative slope composed of incremental waves.	64
23.	Schematic diagram of the front of a square wave, with the x-dimension greatly magnified.	72

#### LIST OF TABLES

<u>Table</u>		<u>Page</u>
1	Dynamic Stress-Strain Curve for Nylon Yarn	14

## NOMENCLATURE

$A$	=	cross-section area of unstretched sample.
$A_{ij}$	=	first time-integral of the repetitive process, corresponding to the lattice point in the $i^{\text{th}}$ column and the $j^{\text{th}}$ row.
$a_1, a_2, a_3, a_4$	=	constants in a trigometric expression.
$B_{ij}$	=	second time-integral of repetitive process.
$c$	=	velocity of propagation of strain waves.
$c_0$	=	propagation velocity at zero strain.
$E_0$	=	modulus of elasticity at zero strain.
$E_1$	=	coefficient in the expression for $\sigma$ .
$\xi$	=	energy imparted to the string in time $t$ .
$F$	=	force acting on mass $m$ within surface $S$ .
$f(x)$	=	function giving the initial displacement of each particle in a string; also (in Appendix I) an arbitrarily specified initial strain.
$g(x)$	=	function giving the initial velocity of each particle in a string.
$\bar{g}$	=	$g / cv_0$ ; the bar is dropped after the quantity is introduced.
$I_k$	=	input for the $k^{\text{th}}$ step of the repetitive process.
$i$	=	subscript indicating the column number of a lattice point; also an index in the slack-cord model.
$J_k$	=	output of the $k^{\text{th}}$ step of the repetitive process.
$j$	=	subscript indicating the row number of a lattice point.
$K$	=	spring constant of all springs joining the discrete masses.
$k$	=	a repetition index (superscript) in the repetitive process.
$L$	=	length of unstretched sample.
$M$	=	mass of each discrete mass; also, the momentum contained within an arbitrary region.
$m$	=	mass contained by surface $S$ , and experiencing force $F$ .

$n$	=	number of lattice-point intervals in the length $L$ ; also the number of discrete masses in the slack-cord model.
$R$	=	$(c/c_0)^2$ , a given function of the strain $\epsilon$ .
$S$	=	surface containing mass $m$ for which the momentum theorem is derived.
$s$	=	displacement of a particle of material from its unstretched position $x$ , also see $\bar{s}$ .
$\bar{s}$	=	$s/Lv_0$ ; the bar is dropped after the quantity is introduced.
$s_F$	=	total stretch in the transition region.
$T$	=	kinetic energy per unit length of string.
$t$	=	time, also see $\bar{t}$ .
$\bar{t}$	=	$tc/L$ ; the bar is dropped after the quantity is introduced.
$v$	=	velocity of a particle of sample.
$x$	=	distance of a particle of material from the fixed (left) end of the sample, in the unstretched state, also see $\bar{x}$ .
$\bar{x}$	=	$x/L$ ; the bar is dropped after the quantity is introduced.
$y_i$	=	amplitude of vibration of the $i^{\text{th}}$ discrete mass.
$\alpha$	=	the first of two weighting factors used to speed up convergence, also the quantity defined by eq (82).
$\beta$	=	the second of two weighting factors used to speed up convergence; $\alpha + \beta = 1$ .
$\gamma$	=	phase angle in slack-cord model or continuous system.
$\epsilon$	=	normal strain along axis of string, $= \partial s / \partial x$ .
$\lambda$	=	wave length of assumed wave in a continuous medium.
$\rho$	=	density of unstretched material.
$\sigma$	=	normal stress along axis of string.
$\Phi$	=	potential energy per unit length of string.
$\psi$	=	function of $t$ in the continuous system.
$\Omega$	=	$2\pi/\lambda$
$\omega$	=	angular frequency in the discrete-mass problem.

## ABSTRACT

An integral-equation, successive-substitution solution for the propagation of strain waves in a linear elastic material, developed in an earlier paper, is here extended to non-linear materials without creep (time-independent materials). A previously-published problem, based on experimental data for nylon string is solved by the successive-substitution method and compared with the earlier solution obtained by the method of characteristics. The agreement of the two solutions is at first good, but in the region where the stress-strain curve of the nylon is concave upward a tendency to oscillation builds up and the solution eventually oscillates. This occurs in approximately the same region where shock waves can appear in a method-of-characteristics solution. Oscillations in the strained portion of a string are examined theoretically and it is shown that the energy of standing waves can be used to obtain energy conservation. Although the successive-substitution method generates oscillations when it has to deal with a discontinuity in strain, the possibility that oscillations also actually exist as a means of conserving energy cannot be ruled out.

## 1. Introduction

The present report is the second in a series of three that describe the results of mathematical and experimental investigations of the response of polymers to tensile impact. Professor Crout is responsible for the mathematical developments and Mr. Pilsworth is responsible for the experimental data. The situation when the work was undertaken and the reasons for adopting the present approach are discussed at length in the first report [1] (subsequently referred to as Report I) of this series and will be only briefly summarized here.

In Report I the mathematical method was developed and was applied to the problem of strain propagation in a linear material having no creep or other time dependence. Since the solution of this problem is known, and since the new method gives the correct solution, its validity for this case has been established, and by inference the new method can also be expected to give correct answers to related problems for which satisfactory solutions have not been available.

The present report extends the mathematical analysis to include non-linear, time-independent materials and in the third report non-linear, time-dependent materials will be treated. The subject of the present report (non-linear, time-independent materials) has been treated by the method of characteristics by Von Karman and Duwez [2], by G. L. Taylor [3], and by Rachmatulin [4]. Hence the mathematical results of the present report are not presented as a new solution but rather as a new method of obtaining a known solution and as the next logical step toward solving the problem of strain propagation in a non-linear, time-dependent material. Since the present method furnishes an alternative to the method of characteristics, a comparison of the results obtained in solving a particular problem by the two methods is given.

## 2. Summary of Report I

A satisfactory theory of the response of materials to loading (tensile impact) should permit us to calculate the stresses and strains in a material, for any rate of loading, from a few fixed properties or parameters of the material. There is no satisfactory theory that will do this. Linear viscoelastic theory is adequate for small strains but not for large strains. Stress-strain curves plus creep or relaxation

data are satisfactory for large or small strains, if the strain rate is very low. The method of characteristics is adequate for non-linear time-independent materials, but as has been shown by Pilsworth and Hoge [5], by Pilsworth [6], and by Smith and Fenstermaker [7] it is inadequate for materials such as Nylon and rubber.

In Report I a new method of handling the problem of strain-wave propagation was developed, in which the relevant partial differential equations are transformed into integral equations. The integral equations are then solved by the method of successive substitutions. Referring to the system shown in Fig. 1, the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

was derived for strain propagation. Here  $x$  is the position of a particle of material in the unstrained state,  $s$  is the displacement of the same particle from its original position, and  $c$  is the wave velocity. Other quantities are defined in the table of nomenclature. By integrating eq (1) twice, the equation

$$u = f(x) + t g(x) + c^2 \int_0^x \int_0^t \frac{\partial^2 u}{\partial x^2} dt dt \quad (2)$$

was obtained; this equation is valid when  $c$  is constant (a linear material has  $c$  constant). The function  $f(x)$  specifies the initial displacements and  $g(x)$  specifies the initial velocities; except at the impacted end of the string these functions are usually zero in practical problems.

Equation (2) was simplified by putting  $f$  and  $g = 0$  and introducing three new normalized variables:

$$\bar{u} = \frac{u}{L v_0}, \quad \bar{t} = \frac{ct}{L}, \quad \bar{x} = \frac{x}{L} \quad (3)$$

in which  $v_0$  is the velocity of the impacted end of the string.

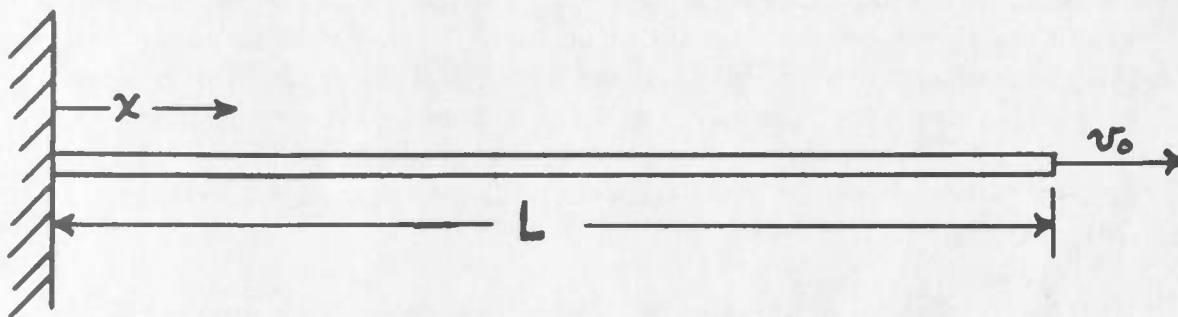


Fig. 1 String subjected to tensile impact.

The result of these substitutions (see Report I, p 36) was to reduce eq (2) to the form

$$s = \int_0^x \int_0^x \frac{\partial^2 s}{\partial x^2} dt dt \quad (4)$$

where for convenience the bars have been dropped from the new variables. The velocity  $c$  does not appear in eq (4); this equation refers, in fact, to the problem in which  $L = 1$ ,  $c = 1$ , and  $v = 1$ . When a solution of eq (4) has been obtained, the results can be applied to actual problems by using eq (3). The method devised for evaluating this integral was to accept a lattice of points in the  $(x, t)$  plane, to specify the values of  $s$  along three borders of this lattice, and to find the values of  $s$  at the remaining points by an iterative procedure based on eq (4). To find values of the integrand the three-point differentiation formula was used, and for the integration the trapezoidal formula was adopted. Values of  $s$  were calculated one row (one value of  $t$ ) at a time by iteration, using the values of the previous row as starting values. When the specified degree of convergence had been reached, the next row of values was calculated.

It was demonstrated that convergence was much accelerated by modifying the successive substitution process. The modification adopted was to multiply the output of the  $k$ th iteration by 0.7 and the input by 0.3, and add the two products to obtain the input for the next iteration. The weighting factors 0.7 and 0.3 appeared to be near the optimum for general use but do not necessarily give the most rapid convergence under all circumstances. It was shown that the use of weighting factors affects only the rate of convergence, not the final result.

Two linear, time-independent problems were solved by digital computer, using the methods summarized above, and the results have been given in Report I. In the first problem the end of the string is set in motion with its full velocity  $v_0$  at  $t = 0$ ; in the second the velocity rises linearly during a finite time interval beginning at  $t = 0$ . Since the exact solution of eq (4) is known, the results obtained by the iterative process could be checked by comparison. For the case of a discontinuous rise in velocity (infinite acceleration at  $t = 0$ ) the values of  $s$  found by iteration were found to

agree quite well with the exact solutions, but when values of the strain  $\epsilon$  ( $= \partial s / \partial x$ ) from the iterative solution were compared with the exact values it was found that the iterative values oscillated above and below the exact values, when plotted versus  $x$ . The oscillations were also present in the values of  $s$ , of course, but were very much less noticeable in  $s$  than in  $\partial s / \partial x$ . The oscillations were largest at the values of  $x$  nearest to the discontinuity of the exact solution. It is clear that the iterative process cannot give results that reproduce a discontinuity exactly. This had been anticipated. In adjusting to the discontinuity the iterative results were found to be first too low, then too high, and so on, but as the discontinuity (wave front) traveled away from any particular point, the amplitude of the oscillations near that point was found to decrease.

The case of a continuous rise in velocity (finite acceleration) was treated because it corresponds more nearly to the situation in actual experiments, and because it was expected that the iterative solution could follow the exact solution in this case with less tendency to be thrown into oscillation than in the case of infinite acceleration. The results justified this expectation. Figure 2, reproduced from Report I, shows the strains calculated from the exact solution (solid line) and from the iterative solution (dots) when the velocity of the impacted end of the string rises linearly from 0 to  $v_0$  as  $t$  increases from 0 to 0.5. (Since the  $t$  used in the calculations is the normalized variable,  $t = 0.5$  is the time at which the leading edge of the strain wave has reached the middle of the string.) The oscillations of the solution obtained by the successive-substitution method are noticeable but not large.

It is important that oscillations caused by discontinuities should not destroy the validity of the successive-substitution method. Hence the effect of other factors on the oscillations due to discontinuities was examined. Since the eventual object of the present work is to be able to calculate the behavior of a non-linear, time-dependent material, the effect of non-linearity was considered. When the stress-strain curve is concave downward, the slope of a strain wave decreases in steepness, and hence the tendency toward oscillations is reduced; but when the curve is concave upward the strain wave increases in steepness so that there is a possibility that a shock wave

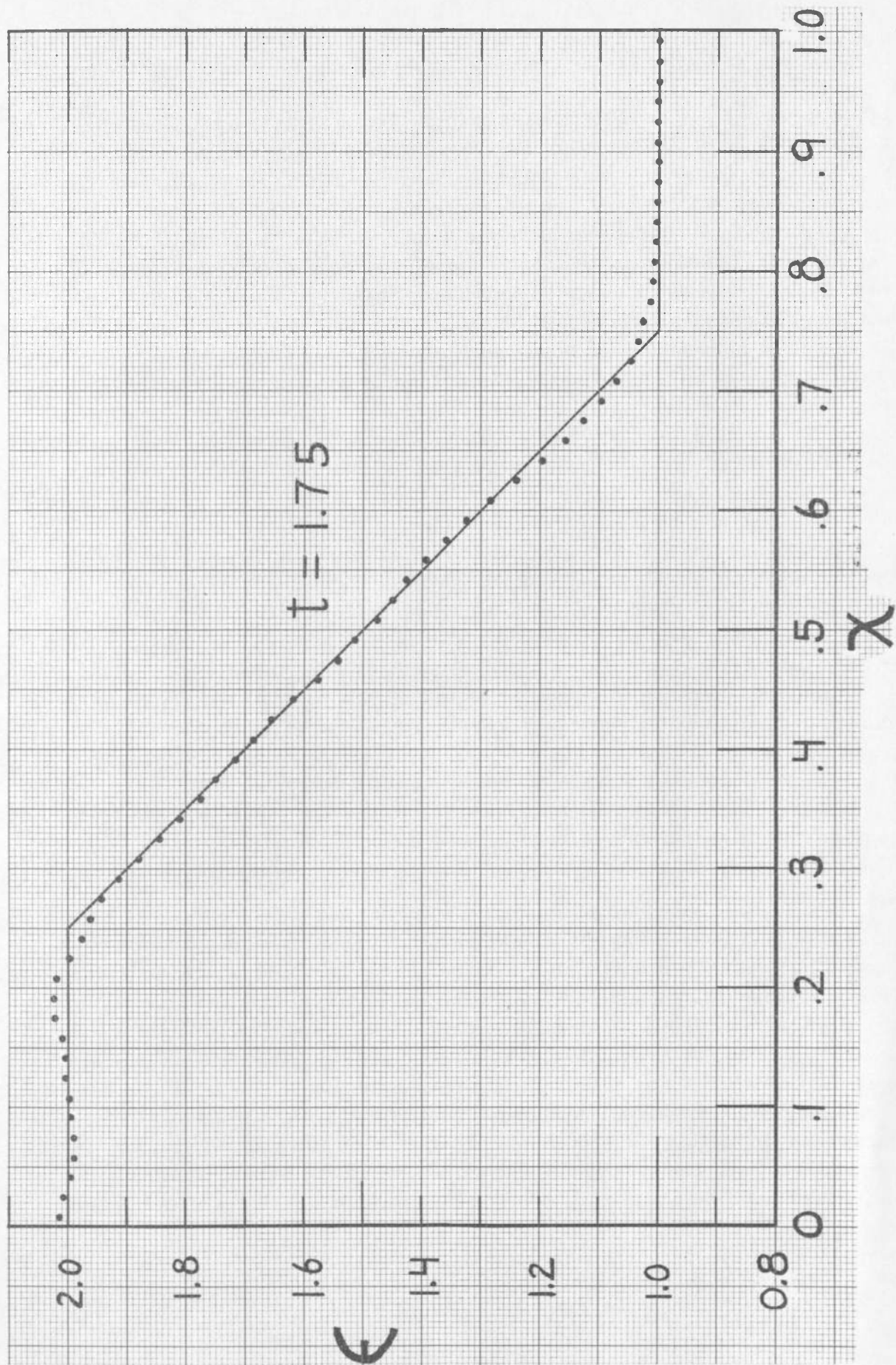


Fig. 2. Strain for a linear time-independent model as obtained from digital computer calculations for  $n = 60$  with  $j = 105$  ( $t = 1.75$ ). The velocity of the impacted end increased linearly to  $v_0$  at  $t = 0.5$ .

will form. Since a shock wave is a discontinuity it will increase the tendency toward oscillations in the iterative solution. Time-dependence of the material will tend to delay a part of the response and hence will tend to destroy discontinuities and reduce the tendency toward oscillations. The conclusion reached in Report I was that the modified successive substitution procedure was satisfactory; and that the oscillations relative to the exact solution could be troublesome but would probably not be serious, except possibly where the stress-strain curve was concave upward and shock waves were mathematically predicted.

### 3. Extension of the Solution to the Non-Linear Case.

The strain at any point in the string is  $\epsilon = \partial s / \partial x$ .  
In the non-linear case

$$\sigma = \sigma(\epsilon), \quad (5)$$

and Newton's law yields the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{\partial \sigma}{\partial x}. \quad (6)$$

The integral equation is

$$u(x, t) = t g(x) + \frac{1}{\rho} \int_0^t \int_0^t \frac{\partial \sigma}{\partial x} dt dt \quad (7)$$

where

$$\left. \begin{aligned} u(x, 0) &= f(x) = 0 \\ \text{and } \left( \frac{\partial u}{\partial t} \right)_{t=0} &= g(x). \end{aligned} \right\} \quad (8)$$

Since

$$\frac{\partial \sigma}{\partial x} = \sigma'(\epsilon) \frac{\partial \epsilon}{\partial x} = \sigma'(\epsilon) \frac{\partial^2 u}{\partial x^2} \quad (9)$$

and since the wave velocity  $c$  is

$$c = \sqrt{\frac{\sigma'(\epsilon)}{\rho}}, \quad (10)$$

it follows that (7) may be written

$$u(x, t) = t \cdot g(x) + \int_0^t \int_0^t c^2 \frac{\partial^2 u}{\partial x^2} dt dt. \quad (11)$$

It should be noted that  $c$  is now a function of  $\epsilon$ , which is  $\partial s / \partial x$ . Let us place

$$\bar{u} = \frac{u}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{c_0 t}{L}, \quad \bar{g}(\bar{x}) = \frac{g(x)}{c_0} \quad (12)$$

where

$$c_0 = \sqrt{\frac{\sigma'(0)}{\rho}}, \quad (13)$$

which is the wave velocity for zero strain.

Substituting in (11) we obtain

$$L \bar{u} = \frac{L \bar{t}}{c_0} c_0 \bar{g} + \frac{L^2}{c_0^2 L} \int_0^{\bar{t}} \int_0^{\bar{t}} c^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} d\bar{t} d\bar{t}, \quad (14)$$

or

$$\bar{u}(\bar{x}, \bar{t}) = \bar{t} \bar{g}(\bar{x}) + \int_0^{\bar{t}} \int_0^{\bar{t}} \left(\frac{c}{c_0}\right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} d\bar{t} d\bar{t}. \quad (15)$$

In the present case

$$\bar{g}(\bar{x}) = 0 \quad 0 \leq \bar{x} \leq 1, \quad (16)$$

it being noted that this relation applies at the right end (no discontinuity in velocity).

Equation (15) thus reduces to

$$\bar{u}(\bar{x}, \bar{t}) = \int_0^{\bar{x}} \int_0^{\bar{t}} \left(\frac{c}{c_0}\right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} d\bar{t} d\bar{t}. \quad (17)$$

Here  $c$  is a function of  $\epsilon$ ; however, it follows from (12) that

$$\epsilon = \frac{\partial u}{\partial x} = \frac{L}{L} \frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial \bar{u}}{\partial \bar{x}} = \bar{\epsilon}; \quad (18)$$

hence

$$c(\epsilon) = c(\bar{\epsilon}) = c\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right). \quad (19)$$

It is now evident that the technique involved in the present report differs from that in Report I only in that the factor  $(c/c_0)^2$  must be included in the integrand.

Finally we note that in addition to (16) we have

$$\left. \begin{aligned} \bar{u}(0, \bar{t}) &= 0 & 0 \leq \bar{t} \\ \bar{u}(\bar{x}, 0) &= 0 & 0 \leq \bar{x} \leq 1 \\ \bar{u}(1, \bar{t}) &= \text{a given function of } \bar{t}; \end{aligned} \right\} \quad (20)$$

also in regard to velocity

$$v = \frac{\partial u}{\partial t} = \frac{L c_0}{L} \frac{\partial \bar{u}}{\partial \bar{t}} = c_0 \frac{\partial \bar{u}}{\partial \bar{t}} = c_0 \bar{v}. \quad (21)$$

#### 4. Description of the Repetitive Process When Applied to the Non-Linear String.

The following set of instructions for programming the successive repetition process for use in connection with a digital

computer was devised. A new quantity

$$R = R(\epsilon) = \left( \frac{\sigma}{\sigma_0} \right)^2 = \frac{\sigma'(\epsilon)}{\sigma'(0)} \quad (22)$$

not used in Report I is needed. The weight factors  $\alpha = .3$ ,  $\beta = .7$ , as described in Report I, p 51, are again used.

Our problem is to determine the values of the quantity  $s_{ij}$  at the points of the square lattice shown in Fig. 3. The index  $i$  takes the values  $0, 1, 2, 3, \dots, n$ ; and  $j$  takes the values  $0, 1, 2, 3, \dots$ . Since  $n$  is given, the range of values taken by  $i$  is bounded. The range of values taken by  $j$ , however, has no upper bound except one which will be assigned later from practical considerations.

The values of  $s_{ij}$  are given for those points which lie on the border, thus

$$\left. \begin{array}{ll} s_{0j} = 0 & j = 0, 1, 2, 3 \dots j \\ s_{i0} = 0 & i = 0, 1, 2, 3 \dots m; \\ s_{mj} \text{ is given} & j = 0, 1, 2, 3 \dots \end{array} \right\} \quad (23)$$

The values of  $s_{ij}$  at the other points are determined a row at a time - that is for  $j$  fixed and  $i = 1, 2, 3, \dots, (n-1)$ . The values of  $j$  are taken in the order  $1, 2, 3 \dots$ . In view of this it will evidently be sufficient to describe how to obtain the values of  $s_{ij}$  for the  $j^{\text{th}}$  row when the values for all of the preceding rows are known.

The values of  $s_{ij}$  for the  $j^{\text{th}}$  row are obtained as the limits gotten, respectively, by applying repeatedly a process shortly to be described. At the beginning of the  $k^{\text{th}}$  repetition we have the quantities

$$s_{1j}^{(k)}, s_{2j}^{(k)}, s_{3j}^{(k)}, \dots, s_{m-1,j}^{(k)}, \quad (24)$$

as approximations to the quantities

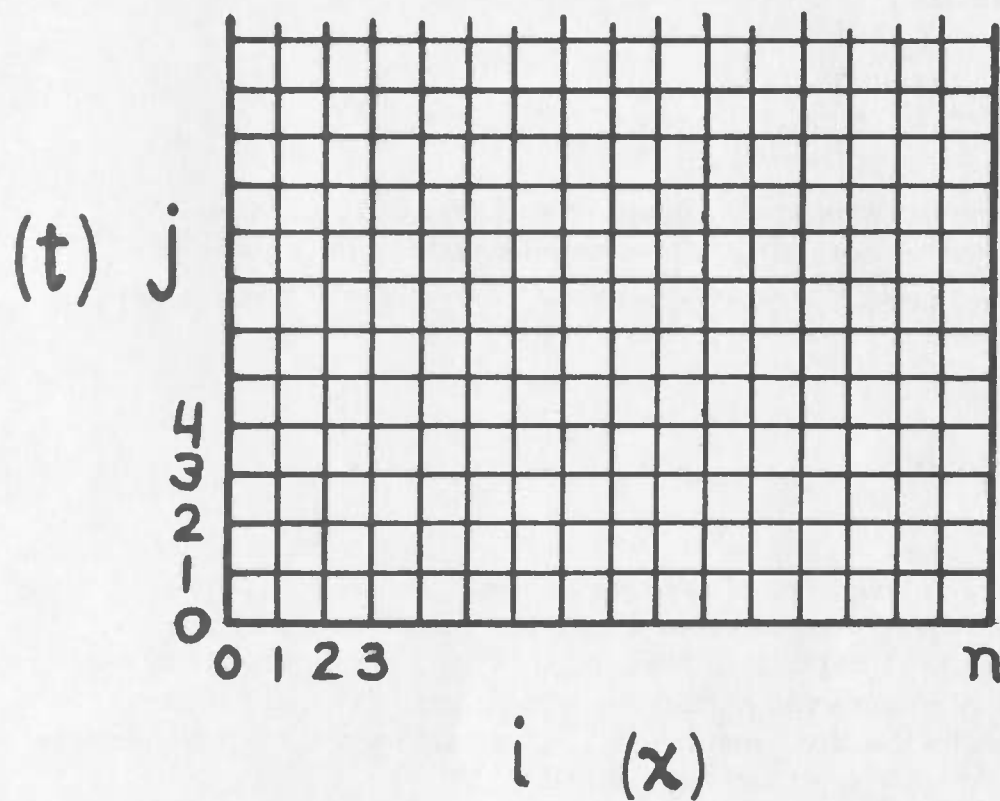


Fig. 3. Lattice of points at which calculations are to be made.

$$s_{1j}, s_{2j}, s_{3j}, \dots, s_{m-1,j} \quad (25)$$

respectively. Here k is a superscript, not an exponent. When the process is applied starting with the quantities (24), the quantities

$$s_{1j}^{(k+1)}, s_{2j}^{(k+1)}, s_{3j}^{(k+1)}, \dots, s_{m-1,j}^{(k+1)} \quad (26)$$

are obtained as approximations to (25), respectively. The sequence of repetitions is started by taking the values of  $s_{ij}$  for the  $(j-1)^{\text{th}}$  row as approximations to the values for the  $j^{\text{th}}$  row, respectively, thus

$$s_{1j}^{(1)} = s_{1,j-1}, s_{2j}^{(1)} = s_{2,j-1}, \dots, s_{m-1,j}^{(1)} = s_{m-1,j-1}. \quad (27)$$

These values are, of course, known. The sequence of repetitions is terminated when, for sufficiently large k, the output (26) duplicates the input (24) to the required accuracy. The common values which then compose (24) and (26) are taken as the final result (25). The values of  $s_{ij}$  which compose the  $j^{\text{th}}$  row are thus determined.

Finally we shall describe the process by which we start with (24) and obtain (26). During the course of the calculations we compute and record, in addition to the quantities  $s_{ij}$ , the auxiliary quantities  $\epsilon_{ij}$ ,  $R_{ij}$ ,  $A_{ij}$ , and  $B_{ij}$ . The quantity A is a measure of the first time integral and the quantity B of the second time integral. Both A and B differ somewhat from the corresponding quantities in Report I because they contain R, which was not used in Report I. The four auxiliary quantities do not exist at the points of the left border of the lattice, for which  $i = 0$ , or the right border, for which  $i = n$ . Along the lower border, where  $j = 0$ ,  $\epsilon_{ij}$ ,  $A_{ij}$ , and  $B_{ij}$  are defined to be zero, and  $R_{ij}$  to be unity. As is the case with  $s_{ij}$  the auxiliary

quantities  $\epsilon_{ij}$ ,  $R_{ij}$ ,  $A_{ij}$ , and  $B_{ij}$  are obtained one row at a time, are known in the first  $(j-1)$  rows, and are obtained in the  $j^{\text{th}}$  row as the limits approached by the quantities  $\epsilon_{ij}^{(k)}$ ,  $R_{ij}^{(k)}$ ,  $A_{ij}^{(k)}$ ,  $B_{ij}^{(k)}$ , respectively, during the course of the successive repetitions. The desired basic process is now described by the relations

$$\epsilon_{ij}^{(k)} = \frac{n}{2} \left( s_{i+1,j}^{(k)} - s_{i-1,j}^{(k)} \right), \quad i = 1, 2, \dots, (n-1) \quad (28)$$

$$R_{ij}^{(k)} = R(\epsilon_{ij}^{(k)}), \quad i = 1, 2, \dots, (n-1) \quad (29)$$

$$A_{ij}^{(k)} = A_{i,j-1} + R_{i,j-1} \left[ \frac{1}{2} (s_{i-1,j-1} + s_{i+1,j-1}) - s_{i,j-1} \right] + R_{ij}^{(k)} \left[ \frac{1}{2} (s_{i-1,j}^{(k)} + s_{i+1,j}^{(k)}) - s_{ij}^{(k)} \right] \quad i = 1, 2, \dots, (n-1) \quad (30)$$

$$B_{ij}^{(k)} = B_{i,j-1} + \frac{1}{2} (A_{i,j-1} + A_{ij}^{(k)}), \quad i = 1, 2, \dots, (n-1) \quad (31)$$

$$s_{ij}^{(k+1)} = .3 s_{ij}^{(k)} + .7 B_{ij}^{(k)}, \quad i = 1, 2, \dots, (n-1) \quad (32)$$

Starting with the given quantities  $s_{ij}^{(k)}$ ,  $j$  being fixed and  $i$  taking the values  $1, 2, \dots, (n-1)$ , we compute the quantities  $\epsilon_{ij}^{(k)}$  using (28); then the quantities  $R_{ij}^{(k)}$  using (29). The function  $R(\epsilon)$  is given, it is

$$R(\epsilon) = \left[ \frac{\text{wave velocity at strain } \epsilon}{\text{wave velocity at strain } 0} \right]^2, \quad (33)$$

and is expressed in a form which is convenient for machine calculation— an algebraic polynomial for example. After  $R$  has been calculated or otherwise found, the quantities  $A_{ij}^{(k)}$  are computed using (30); then the quantities  $B_{ij}^{(k)}$  using (31); and finally the quantities  $s_{ij}^{(k+1)}$  using (32). This process is now repeated, and so on.

## 5. Results of Digital Computer Calculations for a Non-Linear Model

Experimental measurements, made on Nylon by Pilsworth, have been previously published [5,6]. Some of the published data had been analyzed by the method of characteristics, and it was decided that the results of this analysis would be convenient for comparison with results obtained by the present method of successive substitutions. In the earlier investigation, several different stress-strain curves were used for method-of-characteristics calculations; the curve that gave the best results is shown in Fig. 4. It was decided to use this curve as the basis of the comparison.

Table 1 contains the values read from the curve. For reasons to be discussed later it was impossible to use the iterative procedure with these stress-strain values at the highest strains, and in order to obtain an iterative solution at the highest strains the upper part of the stress-strain curve was simply replaced by a straight line. This line is shown dashed in Fig. 4 and the corresponding values are given in parentheses in Table 1.

TABLE 1

### Dynamic Stress-Strain Curve for Nylon Yarn

<u><math>\epsilon</math></u>	<u><math>\sigma</math>, gms/den</u>	<u><math>c</math>, cm/ms</u>
0	0	240
.01	.55	192
.02	.91	172
.03	1.22	159
.04	1.50	153
.05	1.76	150
.06	2.02	154
.07	2.35	175
.08	2.75	220
.09	3.44	314
.10	4.95	408
.11	7.33(6.84)	502(408)
.12	10.75(8.73)	596(408)

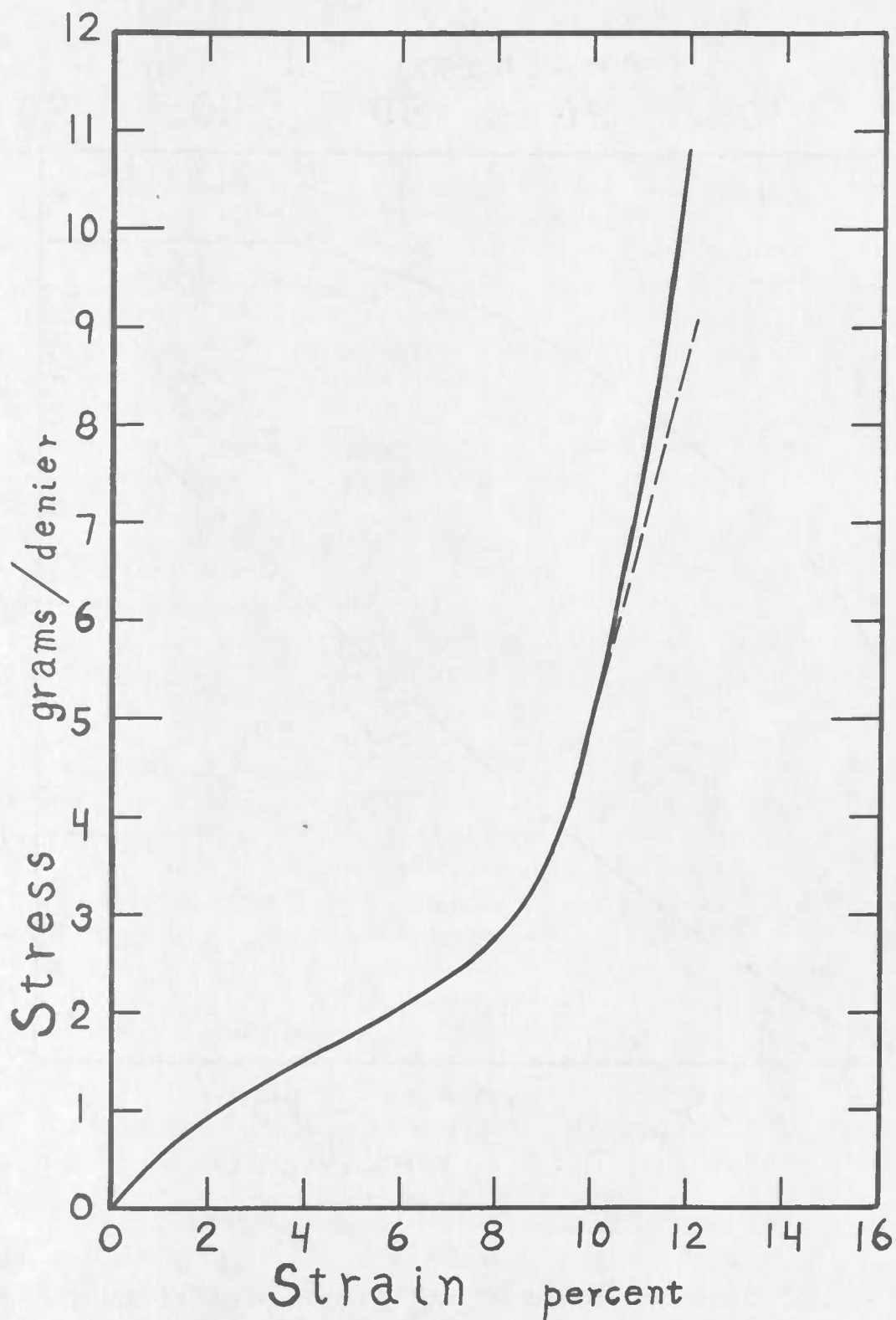


Fig. 4. Dynamic stress-strain curve for nylon yarn.

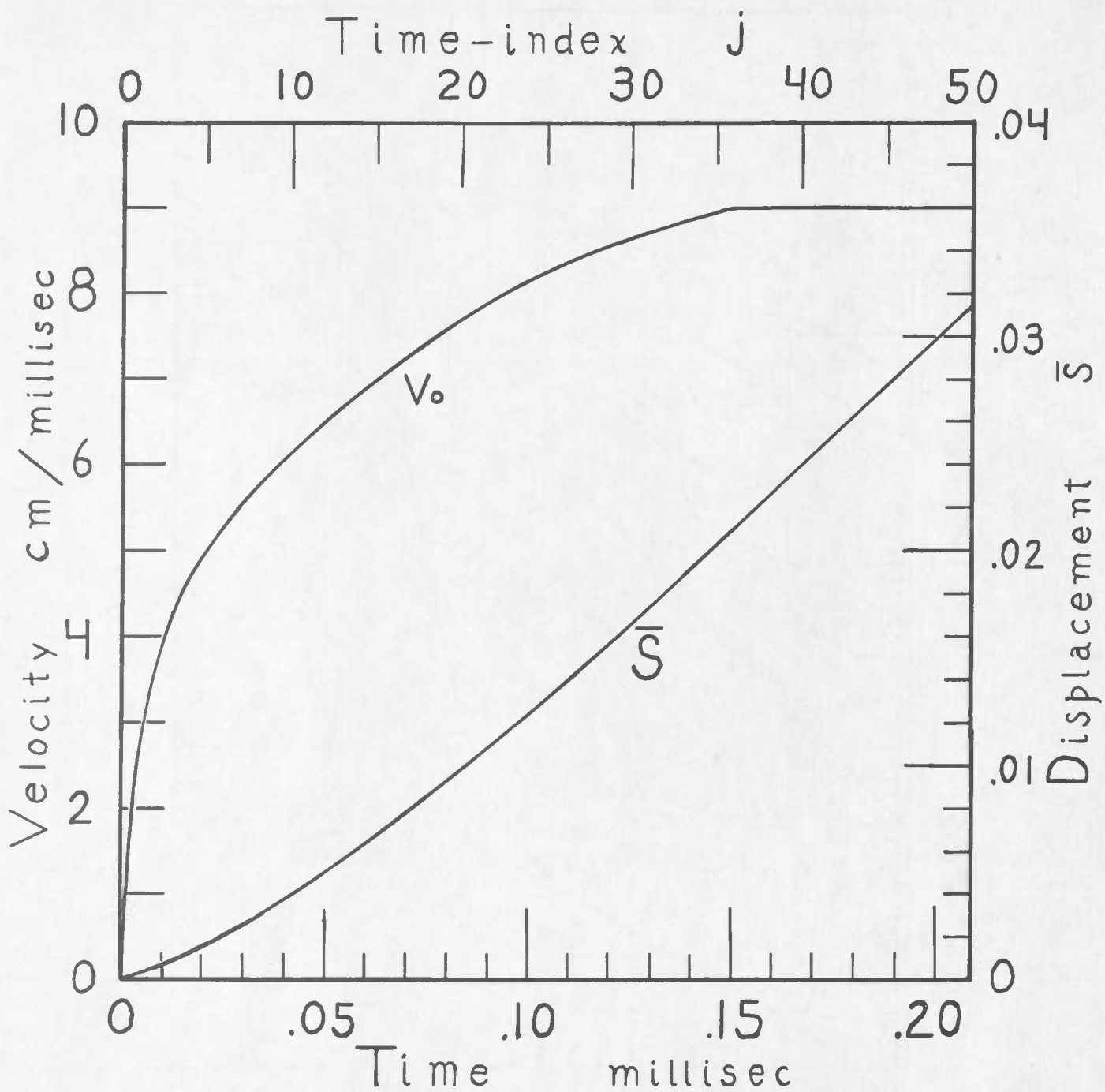


Fig. 5. Displacement and velocity of impacted end of string.

The problem that had been previously solved by the method of characteristics, and which is now solved by the method of successive substitutions, was that of a 50 cm length of Nylon string with one end fixed and the other impacted by a slider that accelerated it from rest to a velocity of 90 m/sec in 0.15 millisecc. The increase in velocity was not linear; the values used were determined from the experimental data. For the method of characteristics the value of  $v_0$  during the period of acceleration must be known; and for the iterative method the values of displacement,  $\bar{s}$ , at the moving end of the string are required. The accepted values of  $v_0$  and  $\bar{s}$  are shown in Fig. 5. The values of  $\bar{s}$  were derived from the previously-used values of  $v_0$  and are consistent with them.

The rise time of the velocity, as determined from the experimental data, was about 0.15 millisecc; this value is typical of the rise times usually observed. The velocity of propagation of small strains, according to Table 1, is 240 cm/millisecc. At this velocity the initial strain will travel 36 cm, or 72% of the length of the string, before the moving end reaches its full velocity.

Figure 6 shows the results of the iterative (successive substitution) solution of the problem. The graph shows strain,  $\epsilon$ , as a function of position,  $x$ . Each plotted line corresponds to a fixed time, that is, to a selected value of the time-index  $j$ . A square lattice-mesh with  $n = 50$  in the normalized variables was used. In the actual variables this corresponds to an interval in  $x$  of  $L/50 = 1$  cm and in  $t$  of  $L/50 c_0 = 4.1667$  microsecc. In the upper part of Fig. 6 there is evidence of oscillations; these will be discussed later. For the present we shall confine our attention to the lower part of the figure where no oscillations occur.

The previously published [5,6] solution of the problem as obtained graphically by the method of characteristics is shown in Fig. 7. The abscissa is  $x$ , the ordinate is  $t$ , and the straight lines are characteristics; except in the regions where characteristics cross they correspond to lines of constant strain. Since the coordinates of Figs. 6 and 7 are different they cannot be readily compared. To permit a comparison a single

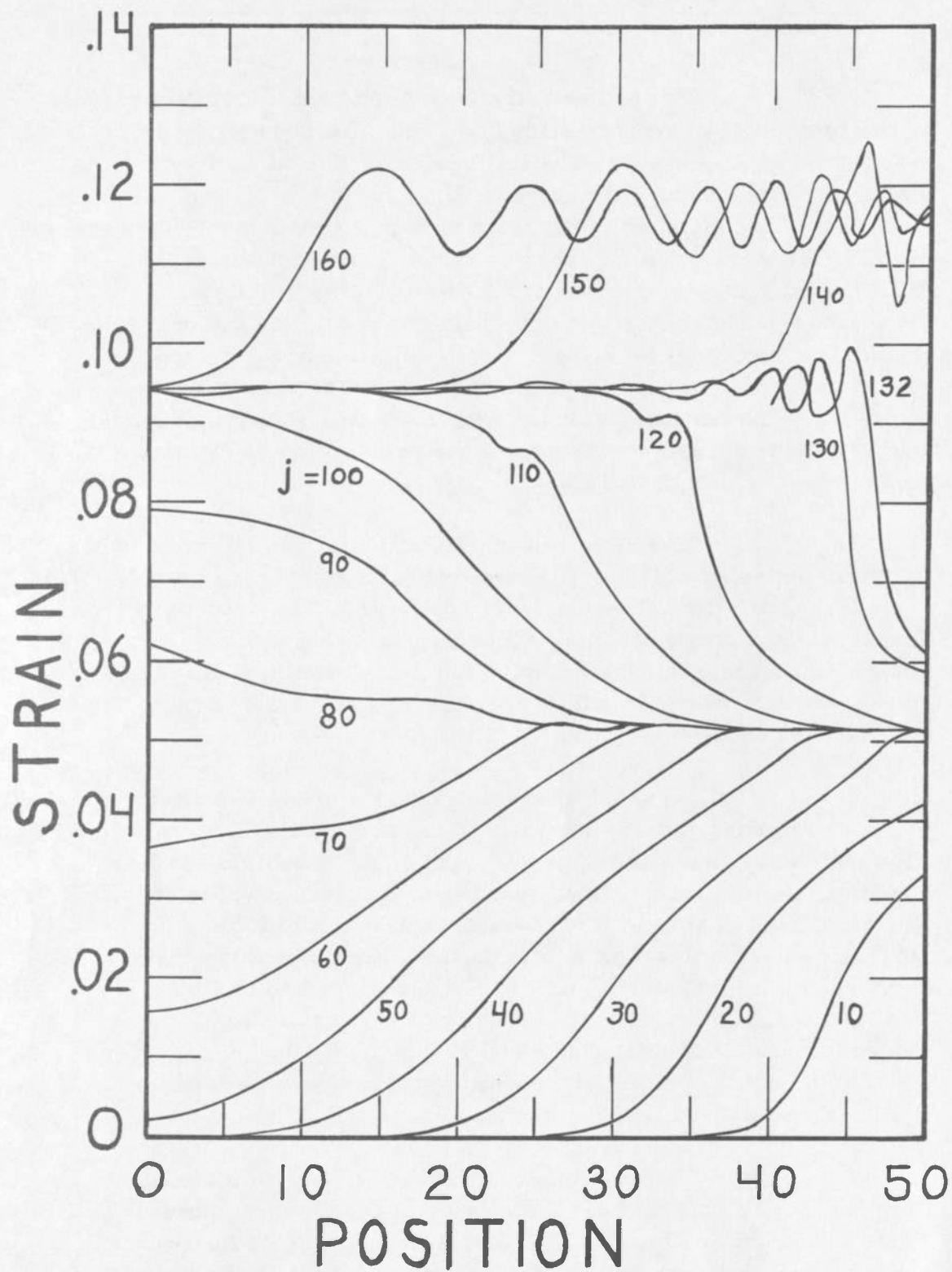


Fig. 6. Strain pattern in nylon yarn calculated by successive substitution for a time-independent model. Time interval between lines is 0.0417 millisec.

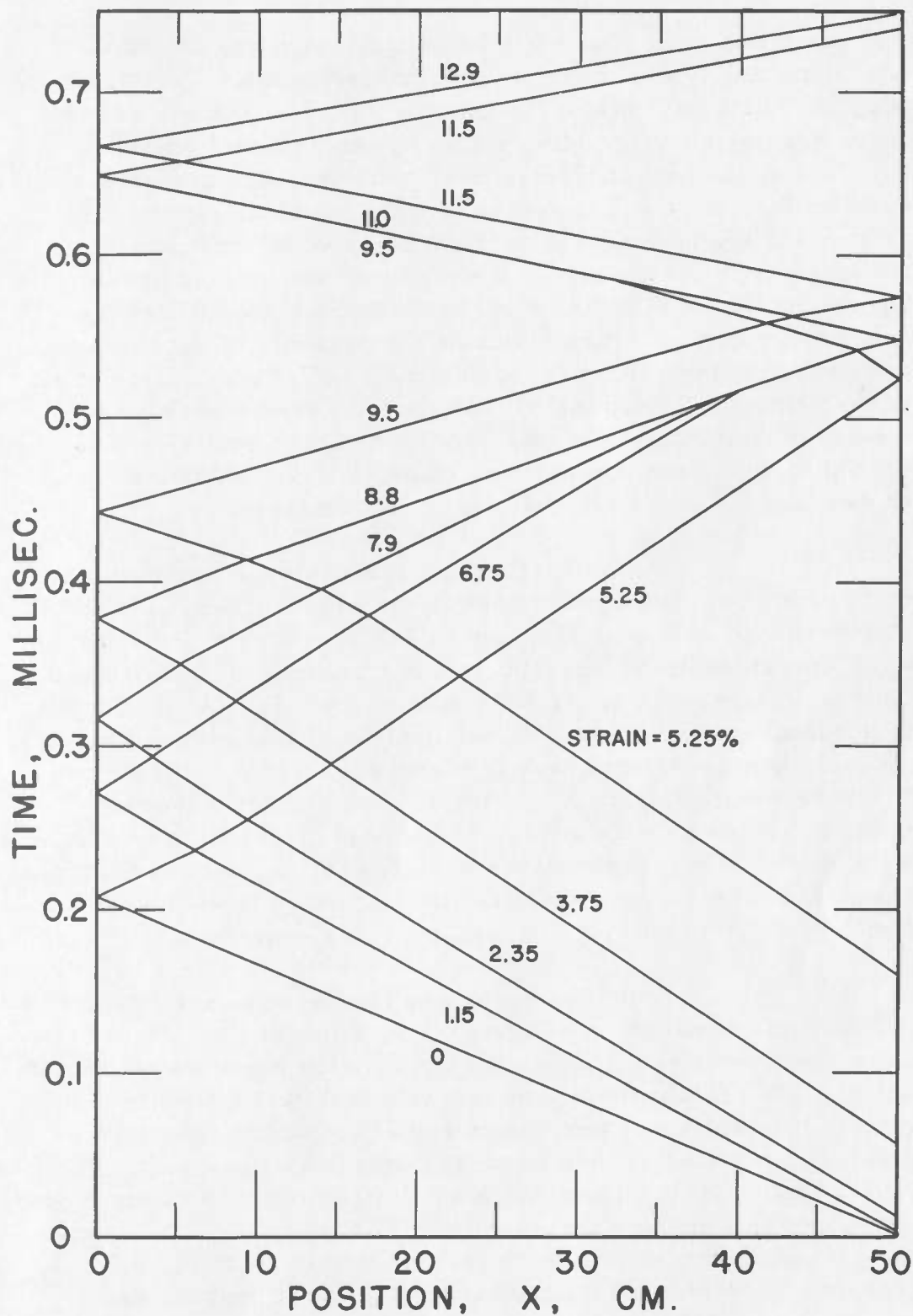


Fig. 7. Strain pattern for nylon yarn calculated by method-of-characteristics.

new graph has been constructed showing  $\epsilon$  as a function of  $x$ , and containing typical results from both solutions. This graph (Fig. 8) contains a few of the lines of Fig. 6 replotted, and in addition some corresponding values (plotted points) obtained from the method-of-characteristics solution. The agreement between the points and the curve is quite satisfactory; the discrepancies are no larger than might be expected when one of the methods is graphical. The two points near the top of the figure that do not fall near a curve lie in the region where the method of characteristics indicated the presence of shock waves and the computer solution gave serious oscillations. Since at all earlier times the iterative method gave the same result as the method of characteristics, its validity has been established for the solution of a non-linear, time-independent problem, so long as oscillations in the solution are not encountered.

We now return to a discussion of oscillations that occur in the iterative method in the region of large strains. The first evidence of oscillations in Fig. 6 occurs at  $j = 110$ , when the strain in the material varies from about 0.051 to about 0.094. Referring to the stress-strain curve, Fig. 4, we see that it is concave upward throughout almost all of this range of strains. When the stress-strain curve is concave upward we may (from the mathematical standpoint at least) expect shock waves to develop. Since a shock wave approximates closely to a discontinuity we should expect it to throw the iterative solution into oscillations; it was shown in Report I that oscillations are generated when the iterative solution attempts to conform to a discontinuity.

It may be asked why the iterative solution does not generate oscillations before  $j = 110$ , since at that point almost all of the sample is within the region in which shock waves should develop. A reasonable explanation of this is that the finite rise time of the velocity of the moving end of the string generated a sloping, not a steep, wave front traveling down the string. Furthermore, the initial portion of the stress-strain curve is concave downward, and this flattens the wave front still more. As soon as the part of the curve that is concave upward is reached, the wave front begins to steepen but only at  $j = 110$  has it become steep enough to generate oscillations.

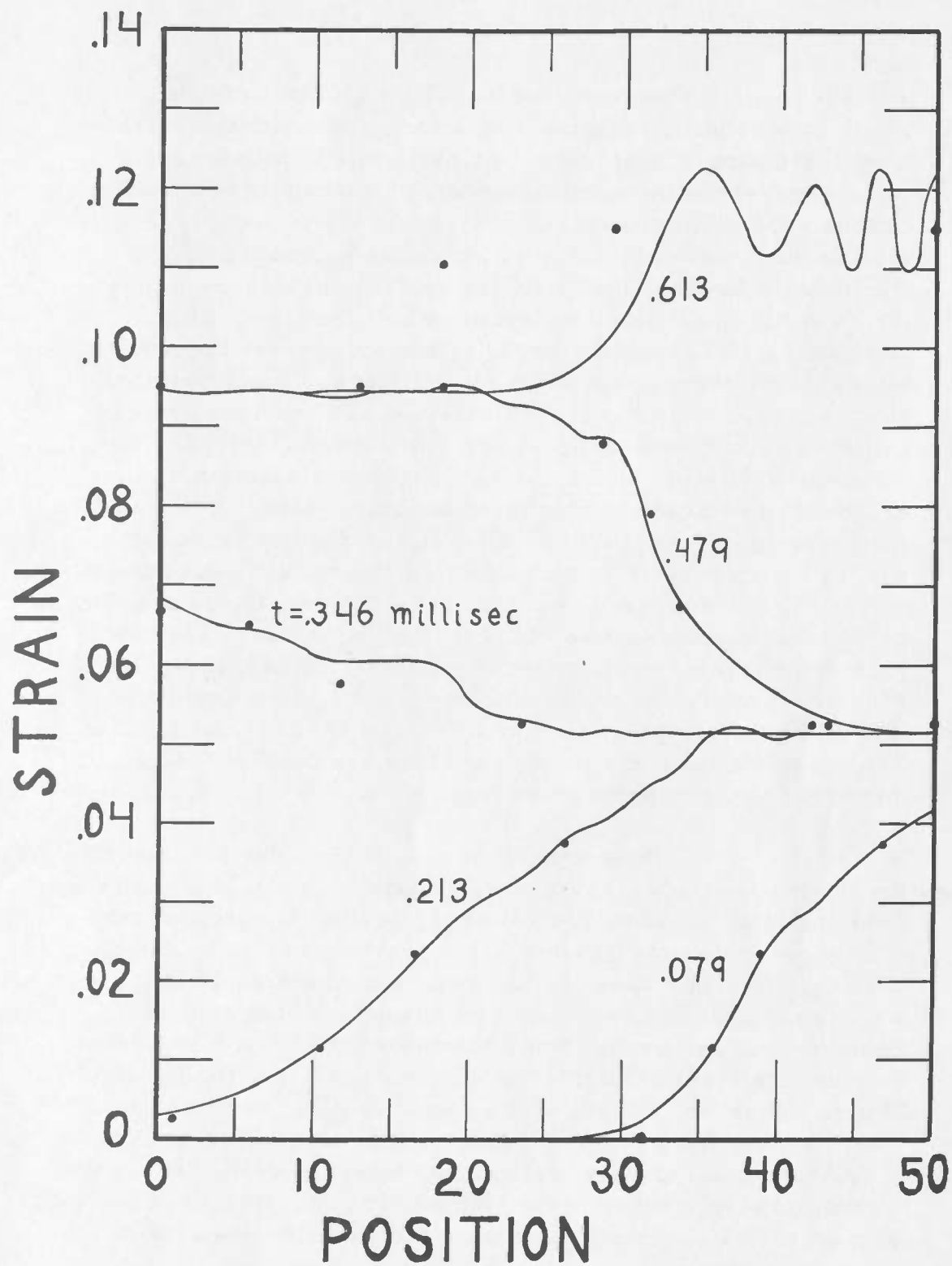


Fig. 8. Strain pattern for nylon yarn calculated by successive substitution (lines) and by method-of-characteristics (points).

Between  $j = 110$  and  $j = 132$  the iterative, oscillatory solution continued to converge, but with the oscillations increasing in amplitude. At  $j = 133$  the oscillations rapidly increased beyond the capacity of the computer. Modifications were then made in the program to try to reduce the oscillations at least to the point where the program could be completed. Decreasing the lattice spacing reduced the wavelength of the oscillations but didn't reduce their amplitude significantly. Varying the error criterion, that is, the maximum allowable difference between the results of two successive substitutions when the values were accepted as converged, was no help. Using a five-point least squares formula instead of a three-point formula to calculate strain from displacement reduced the oscillations somewhat but only delayed the breakdown from row 133 to row 137. Finally, it was decided to modify a part of the stress-strain curve that was being used, the solid curve of Fig. 4. A large section of this curve is concave upward, and a part of this section was replaced by a straight line. The modified part of the stress-strain curve is shown by the dashed line in Fig. 4. Examples of the results computed from the modified stress-strain curve are the curves for  $j = 140, 150,$  and  $160$  in Fig. 6; oscillations are present but they are relatively small and do not cause the solution to diverge.

The substitution of a straight line for a portion of the stress-strain curve that was concave upward was made principally to see what effect it would have on the oscillations. It is suggested by the graphical procedure that must be used in dealing with shock waves by the method of characteristics, in which the states before and after a shock lie at the ends of a chord joining two points on the stress-strain curve. Aside from this the use of the straight-line segment has no particular significance. However, a study of the equations of motion in regions where shocks were possible led us to the conclusion that in some situations actual physical oscillations might be generated, if the model we were studying were sufficiently sophisticated to represent an actual material. This possibility is discussed further in the next section.

## 6. Nature of the Oscillations when the Stress-Strain Curve is Concave Upward

The oscillations that occurred in the solution of the linear case were explained in Report I as computational hunting above and below the true solution, and they were attributed to the inability of the computational technique to conform to discontinuities. So when oscillations are encountered in the solution of the present (nonlinear) case, it is natural to assume that they are also computational. The fact that the oscillations do not become troublesome until the stress-strain curve is concave upward tends to support this assumption, for when the curve is concave upward the front of a strain wave becomes steeper and a shock wave begins to develop. Since a shock wave is for mathematical purposes a discontinuity our computational techniques should have difficulty in conforming to it and oscillations should be expected. We accept this argument as valid; however, it does not appear to be the full explanation of the oscillations observed.

The much greater amplitude of the oscillations in the non-linear case (often exceeding the capacity of the computer) led us to consider whether the oscillations might not be partly computational and partly "real". By "real" we mean that the exact solution of the problem might show oscillations even if the oscillations introduced by our computational techniques were not present.

It should be remembered that the problem whose solution is shown in Fig. 6 is only an approximation to an actual problem. The problem solved in the figure does not contain the time-dependency that we believe is present in actual polymers. Time-dependency tends to smooth out discontinuities and might prevent oscillations from developing in an actual material. We know of no cases where shock waves have actually been produced by tensile impact on polymers. Nevertheless there is some indication that oscillations may be observed experimentally. Much of the data taken in this laboratory appear to show small oscillations in the strain above and below a smooth curve of strain versus position. These oscillations are at the very limit of the sensitivity of our measurements and we have always smoothed them out at an early stage of our analysis.

Petterson, Stewart, Odell and Maheux [8] also observed oscillations in strain patterns. Their strains were produced by transverse impact of bullets on nylon yarns; the oscillations were quite large. These authors believe the oscillations were not the result of any defects in the measurement but were actually present. However, since no explanation of the oscillations was available, they were smoothed out before the data were analyzed.

We shall now present some theoretical considerations on the problem of whether oscillations should occur in the tensile impact of a time-independent material with a stress-strain curve that is concave upward over part of its length.

#### 7. Theoretical Investigations Pertaining to Oscillations. Momentum Theorem.

In what follows we shall need a momentum theorem which enables us to investigate the momentum balance in a region whose shape, size and motion are specified arbitrarily. At time  $t$  let the closed surface  $S$  in Fig. 9 contain a mass  $m$  distributed in any manner, the momentum of this mass being the vector  $M$ , and the vector sum of the external forces acting on this mass being  $F$ . An infinitesimal time  $dt$  later the mass  $m$  has moved, and occupies a volume bounded by the surface  $S_2$ . At this time the momentum has increased by the amount  $dM_2$  given by

$$dM_2 = F dt. \quad (34)$$

The volume whose momentum balance we are investigating coincides at time  $t$  with that bounded by the surface  $S$ . At time  $t$  it hence contains the mass  $m$  and the momentum  $M$ . At time  $t+dt$  this volume has changed in a prescribed manner and is bounded by the surface  $S_1$ . The momentum of the mass contained within it has increased by the amount  $dM_1$ , which may be written

$$dM_1 = dM_1 - dM_2 + dM_2 = (dM_1 - dM_2) + F dt \quad (35)$$

due to (34). But

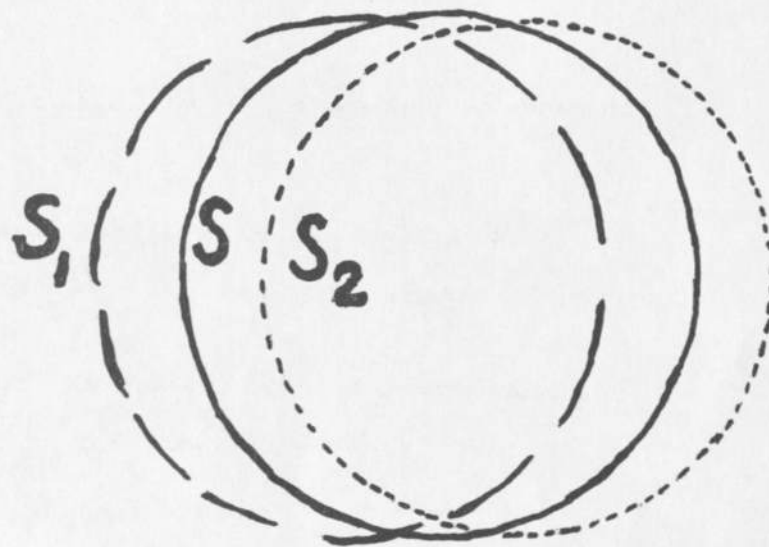


Fig. 9. Closed surfaces for momentum theorem. At time  $t$ ,  $S$  contains a given mass. At  $t + dt$  mass has moved to  $S_2$  and position of  $S$  has moved to  $S_1$ .

$$\begin{aligned}
dM_1 - dM_2 &= (\text{momentum in } S_1 \text{ at time } t+dt \\
&\quad - \text{momentum in } S \text{ at time } t) \\
&\quad - (\text{momentum in } S_2 \text{ at time } t+dt \\
&\quad - \text{momentum in } S \text{ at time } t) \\
&= \text{momentum in } S_1 \text{ at time } t+dt \\
&\quad - \text{momentum in } S_2 \text{ at time } t+dt \\
&= (\text{momentum inside of } S_1 \text{ but outside} \\
&\quad \text{of } S_2 \text{ at time } t+dt) \\
&\quad - (\text{momentum inside of } S_2 \text{ but outside} \\
&\quad \text{of } S_1 \text{ at time } t+dt) \\
&= (\text{momentum of mass entering } S_1 \\
&\quad \text{during time interval } dt) \\
&\quad - (\text{momentum of mass leaving } S_1 \\
&\quad \text{during time interval } dt) \\
&= \text{net momentum entering } S_1 \text{ by} \\
&\quad \text{convection in time interval } dt \\
&\quad \text{due to relative motion of } S_1 \\
&\quad \text{and the mass.} \tag{36}
\end{aligned}$$

Substituting in (35) and dividing by  $dt$  we obtain

$$\frac{dM_1}{dt} = F + \text{rate of increase of momentum in } S_1 \text{ due to the motion of } S_1 \text{ relative to the mass.} \quad (37)$$

The second term on the right hand side is of the nature of a convection term. Equation (37) is the desired theorem. Using it we shall derive certain consequences of the assumption that an abrupt increase in strain can be propagated at constant velocity  $c$ .

Propagation of an Abrupt Increase in Strain. Let us suppose that the strain increases abruptly from 0 to  $\epsilon$  at a vertical wave front which travels to the left with a velocity  $c$ , as indicated in Fig. 10. As usual  $x$  indicates distance along the string in its unstretched state. Let us consider the region bounded by the two arbitrarily chosen, stationary dotted lines, which at time  $t$  lie close to the wave front, one on each side, as shown. These lines play the role of  $S_1$  in (37).

The only external force acting is  $\sigma A$ , where  $A$  is the cross section of the (unstretched) string, this being exerted by the string on the right.

In time  $dt$  the wave progresses a distance  $c dt$  to the left. The length  $c dt$  of string thus undergoes a change in strain from 0 to  $\epsilon$ ; hence a length  $c \epsilon dt$  of (stretched) string passes through the right hand boundary (dotted line). Dividing by  $dt$  we obtain for the velocity  $v$  of the string at this boundary

$$v = c \epsilon. \quad (38)$$

Since the position of this boundary was chosen arbitrarily it follows that the velocity is the same and given by (38) at all points of the string to the right of the wave front.

Let  $A$  be the cross-section area of the string. Then in time  $dt$  a mass  $\rho A c dt$  undergoes an abrupt change from velocity 0 to velocity  $c \epsilon$ . The corresponding momentum change is  $c^2 \epsilon \rho A dt$ . It follows that the increase in the momentum contained in the test region bounded by the two dotted lines is

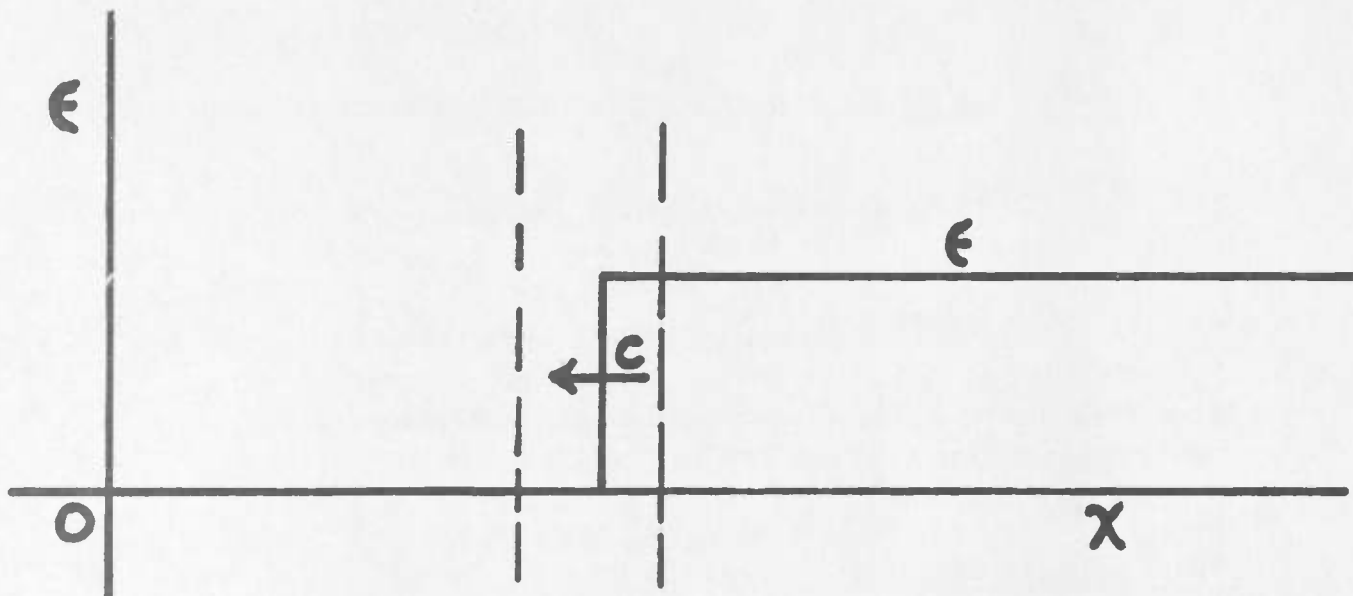


Fig. 10. Strain wave with vertical front.

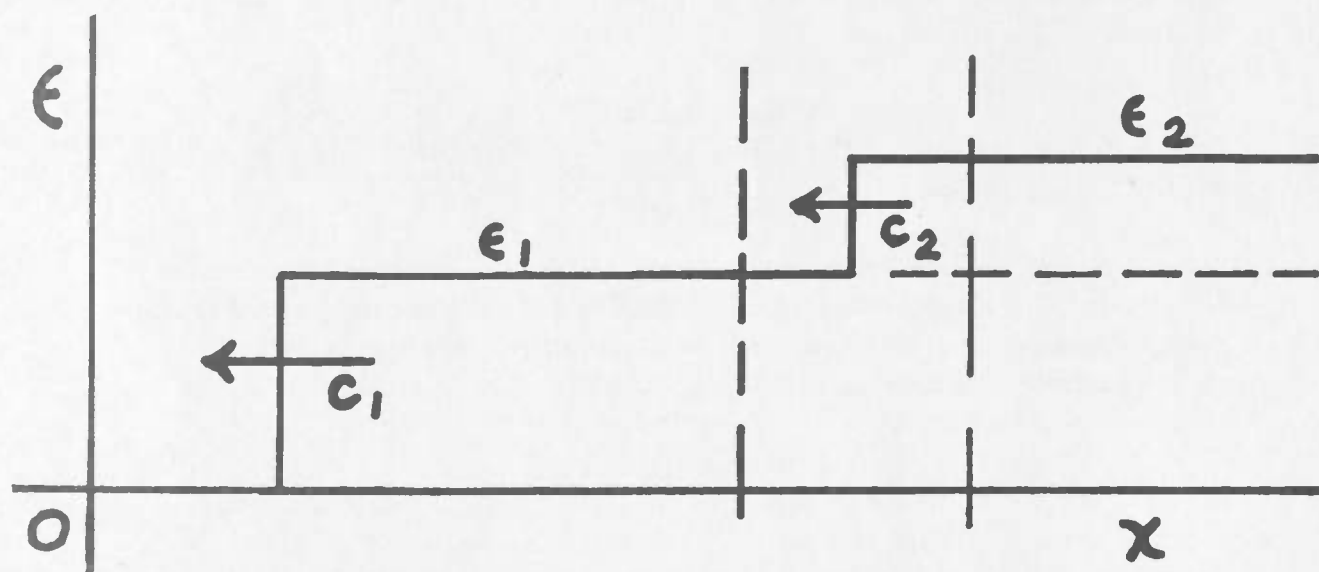


Fig. 11. One verical wave front being overtaken by another.

$$c^2 \epsilon \rho A dt - \text{momentum of mass crossing the right-hand boundary (going to the right).} \quad (39)$$

But (37) states that this quantity is

$$\left. \begin{aligned} F dt + \text{momentum crossing the right-hand boundary going to the left} \\ = F dt - \text{momentum crossing the right-hand boundary going to the right.} \end{aligned} \right\} \quad (40)$$

Equating (39) and (40) gives

$$\left. \begin{aligned} c^2 \epsilon \rho A dt &= F dt = \sigma A dt \\ c^2 \rho \epsilon &= \sigma \\ c &= \sqrt{\frac{\sigma}{\rho \epsilon}}. \end{aligned} \right\} \quad (41)$$

Note. If the boundaries of the test region had been made to move with the string, the convection terms would be missing from (39) and (40); however, since these terms cancel, the result (41) would remain unaltered.

Next let us consider the energy balance of the test region between the dotted lines shown in Fig. 10; however, this time we shall let the boundaries of this region move with the string. In time  $dt$  we have, noting (38) and (41),

$$\left. \begin{aligned} \text{energy input of the} \\ \text{external force } \sigma A &= c \epsilon \sigma A dt \end{aligned} \right\} \quad (42)$$

$$\begin{array}{l} \text{increase in kinetic} \\ \text{energy (at wave front)} \end{array} = \frac{1}{2} \rho A c dt (\epsilon)^2 \quad (43)$$

$$= \frac{1}{2} \rho c \epsilon \sigma A dt. \quad (44)$$

Also

$$\begin{array}{l} \text{increase in potential} \\ \text{(strain) energy in} \\ \text{string (at wave front)} \end{array} = A c dt \int_0^\epsilon \sigma(\epsilon) d\epsilon$$

$$= \rho c \epsilon \sigma_{ave} A dt \quad (45)$$

where  $\sigma_{ave}$  is the average value of  $\sigma(\epsilon)$  over the strain interval 0 to  $\epsilon$ .

The corresponding power equations are obtained by dividing (42), (44), and (45) by  $dt$ . We note that exactly half of the energy input is utilized in increasing the kinetic energy, which leaves an amount  $\frac{1}{2} \rho c \epsilon \sigma A dt$  to balance the increase in potential energy.

In the linear case, where  $\sigma$  is proportional to  $\epsilon$ , it is true that

$$\sigma_{ave} = \frac{1}{2} \sigma$$

and we get a perfect energy balance. If, however, the stress-strain curve is concave downward

$$\sigma_{ave} > \frac{1}{2} \sigma$$

and the available input energy is not sufficient to provide the required increase in potential energy. The abrupt wave front

postulated cannot be maintained. Actually, the top of the wave lags behind the bottom, the wave spreads out, and the front loses its abruptness.

Finally, the case in which we are primarily interested is that in which the stress-strain curve is concave upward, in which case

$$\sigma_{ave} < \frac{1}{2} \sigma$$

and the amount of input energy available is more than enough to provide the required increase in potential energy. The postulated abrupt wave front moving with a constant velocity is hence not in dynamic equilibrium, and cannot be maintained. In an actual problem the possibility exists that mechanical energy may be dissipated as heat. Such a mechanism cannot explain the present situation, however. The laws of classical mechanics must be obeyed, even while kinetic energy is being converted into heat. It may be mentioned, however, that the excess energy might produce oscillations such as those mentioned on page 23. This, of course, infers a modification of our present abrupt form of wave.

#### Effect of the Merging of Two Waves.

Further insight into the nature of the difficulties encountered when the stress-strain curve is concave upward can be obtained by considering two successive abrupt waves, and seeing what happens when one overtakes the other, Fig. 11. Here the two waves were started at different times. In time  $dt$  the two waves progress by amounts  $c_1 dt$  and  $c_2 dt$ , respectively; the string between the two fronts moves a distance  $c_1 \epsilon_1 dt$ , and the string on the right of the second front moves a distance  $[c_1 \epsilon_1 + c_2 (\epsilon_2 - \epsilon_1)] dt$ . Dividing by  $dt$  we obtain the corresponding velocities, thus

$$\left. \begin{aligned} v_1 &= c_1 \epsilon_1 = \text{velocity between fronts} \\ v_2 &= c_1 \epsilon_1 + c_2 (\epsilon_2 - \epsilon_1) = \text{velocity to right of second front.} \end{aligned} \right\} \quad (46)$$

Equation 41 can be applied to give the velocity of the first front, thus

$$c_1 = \sqrt{\sigma_1 / \rho \epsilon_1} \quad \text{where } \sigma_1 = \sigma(\epsilon_1). \quad (47)$$

In order to obtain the velocity of the second front we shall apply the momentum theorem (37) to the test region lying between the two vertical dotted lines in Fig. 11, which lines move with the string. In time  $dt$  the momentum in this region increases by the amount  $\rho A c_2 dt (v_2 - v_1)$ , the convection term is zero, and the total external force is

$$F = (\sigma_2 - \sigma_1) A \quad \text{where } \sigma_2 = \sigma(\epsilon_2). \quad (48)$$

Equation (37) thus becomes

$$\rho A c_2 (v_2 - v_1) = (\sigma_2 - \sigma_1) A$$

or noting (46)

$$\rho c_2^2 (\epsilon_2 - \epsilon_1) = \sigma_2 - \sigma_1$$

$$c_2 = \sqrt{\frac{\sigma_2 - \sigma_1}{\rho(\epsilon_2 - \epsilon_1)}}. \quad (49)$$

We are supposing that the stress-strain curve is concave upward; hence, we see from (47) and (49) that

$$c_2 > c_1. \quad (50)$$

It follows that the second wave is certain to overtake the first. When it does we again have a single abrupt wave front, the strain discontinuity being  $\epsilon_2$ . If this wave proceeds\* with velocity  $c$ , it follows from (41) that

$$c = \sqrt{\frac{\sigma_2}{\rho \epsilon_2}}; \quad (51)$$

also the string velocity immediately to the right of this wave is

$$v = c \epsilon_2 \quad (52)$$

---

\* See Appendix 2 for some comments on an alternative approach, in which the merged wave is assumed to propagate with a velocity lower than  $c$ .

due to (38). The difference quotients which appear under the radicals in the expressions for  $c_1$ ,  $c_2$ , and  $c$  are the slopes of the three chords shown in the stress-strain curve of Fig. 12.

Just before the second wave catches up with the first the velocity of (almost) the entire string to the right of the wave is  $v_2$ . Just after the second wave overtakes the first the velocity immediately to the right of the wave front is  $v$ . The velocity of the entire string to the right of the wave cannot change abruptly from  $v_2$  to  $v$ ; hence, unless these quantities are equal, the merging of the two waves does not take place smoothly, and a tightening or slackening of the string occurs. The corresponding change in stress, and hence strain, constitutes a disturbance which causes a corresponding disturbance to be propagated along the string to the right away from the wave front.

Substituting (47) and (49) in (46), and (51) in (52), we obtain

$$\begin{aligned} v &= \sqrt{\frac{\sigma_2 \epsilon_2}{\rho}}, \\ v_2 &= \sqrt{\frac{\sigma_1 \epsilon_1}{\rho}} + \sqrt{\frac{(\sigma_2 - \sigma_1)(\epsilon_2 - \epsilon_1)}{\rho}}. \end{aligned} \quad (53)$$

Since the stress-strain curve is concave upward it is evident that

$$\sigma_2 > \frac{\epsilon_2}{\epsilon_1} \sigma_1,$$

or

$$\sigma_2 \epsilon_1 > \sigma_1 \epsilon_2.$$

It follows that

$$\begin{aligned} \sqrt{\sigma_2 \epsilon_1} &> \sqrt{\sigma_1 \epsilon_2} \\ 0 &> -(\sqrt{\sigma_2 \epsilon_1} - \sqrt{\sigma_1 \epsilon_2})^2 \\ 0 &> -\sigma_2 \epsilon_1 - \sigma_1 \epsilon_2 + 2\sqrt{\sigma_1 \sigma_2 \epsilon_1 \epsilon_2} \\ \sigma_2 \epsilon_2 + \sigma_1 \epsilon_1 - 2\sqrt{\sigma_1 \sigma_2 \epsilon_1 \epsilon_2} &> \sigma_2 \epsilon_2 + \sigma_1 \epsilon_1 - \sigma_2 \epsilon_1 - \sigma_1 \epsilon_2 \\ (\sqrt{\sigma_2 \epsilon_2} - \sqrt{\sigma_1 \epsilon_1})^2 &> (\sigma_2 - \sigma_1)(\epsilon_2 - \epsilon_1) \\ \sqrt{\sigma_2 \epsilon_2} - \sqrt{\sigma_1 \epsilon_1} &> \sqrt{(\sigma_2 - \sigma_1)(\epsilon_2 - \epsilon_1)} \\ \sqrt{\frac{\sigma_2 \epsilon_2}{\rho}} &> \sqrt{\frac{\sigma_1 \epsilon_1}{\rho}} + \sqrt{\frac{(\sigma_2 - \sigma_1)(\epsilon_2 - \epsilon_1)}{\rho}}. \end{aligned} \quad (54)$$

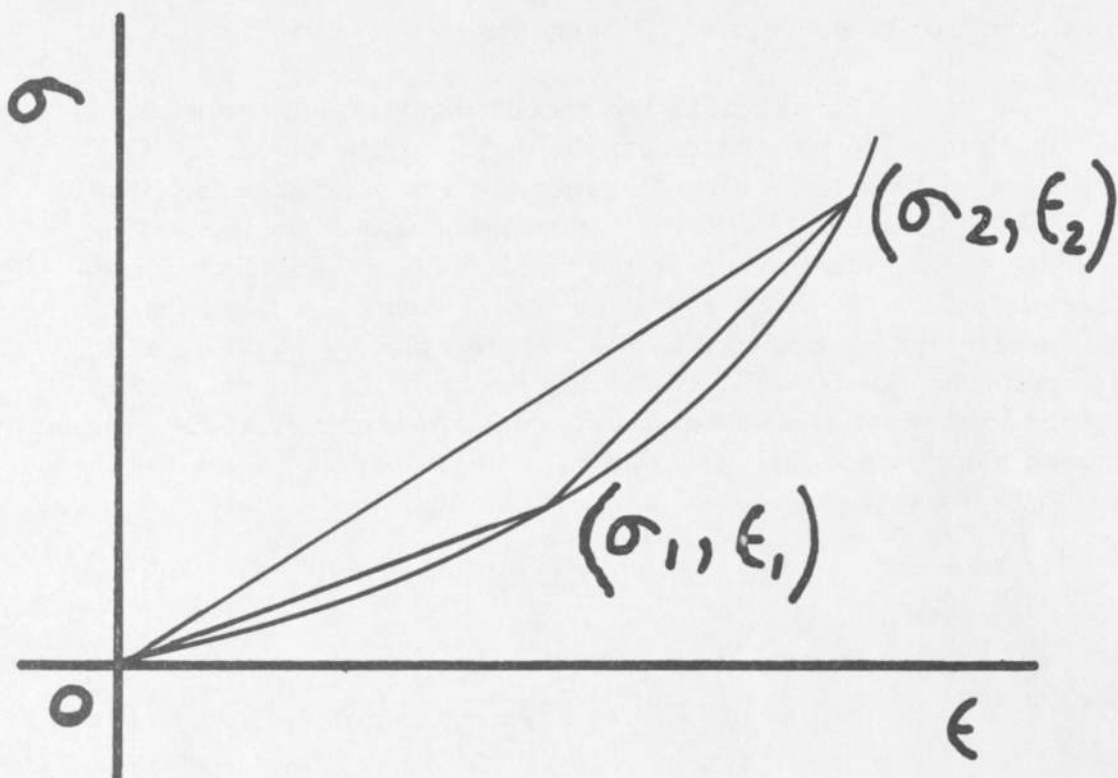


Fig. 12. Stress-strain curve showing chords that determine velocities of vertical wave fronts.

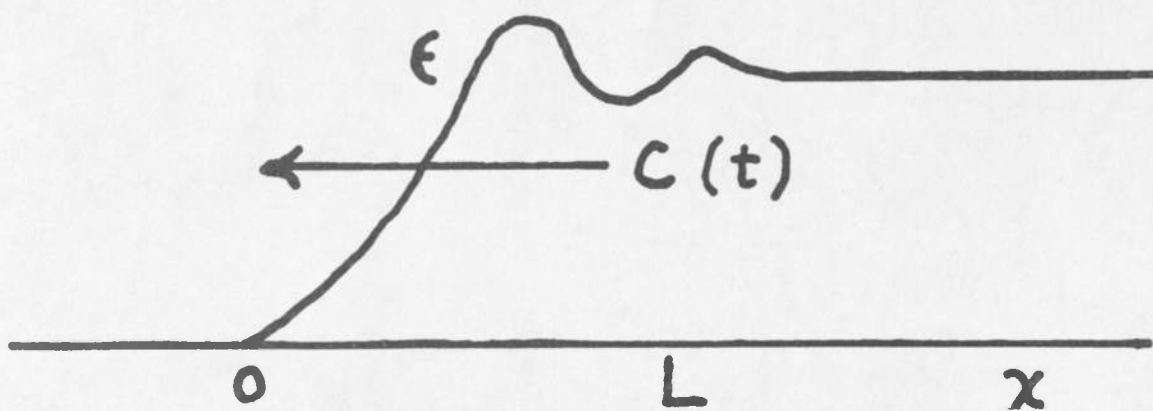


Fig. 13. Wave front of arbitrary but constant shape.

In view of (53) this may be written

$$v > v_2. \quad (55)$$

It follows, as described on page 33, that the merging of the two waves is not smooth, but causes a disturbance which travels along the string in the opposite direction from that of the wave.

Note. The situation portrayed in Fig. 11 can be achieved by using a stress-strain curve composed of two straight lines. One passes through the origin, and is valid for  $0 \leq \epsilon \leq \epsilon_1$ ; the other is steeper, and is valid for  $\epsilon_1 \leq \epsilon \leq \epsilon_2$ .

Effect of a Change of Shape of the Strain-Wave Front. From Fig. 12 and the equations (47), (49), (51), and (55) we saw that when one abrupt strain wave front catches up with another there is a sudden slackening of the string behind the wave front, and a sudden increase in the velocity, or surging forward of the wave.\* This suggests that the wave front loses its abruptness and constancy of velocity. Accordingly let us consider the case of a wave front of constant shape which moves in translation with a velocity  $c(t)$ , no longer required to be constant, Fig. 13.

At time  $t = 0$  let the displacement  $s(x, 0)$  be denoted by  $f(x)$ , thus

$$s(x, 0) = f(x). \quad (56)$$

At present we shall leave  $f(x)$  arbitrary. At time  $t$  this displacement-position curve moves a distance  $\int_0^t c \, dt$  to the left; hence

$$s = f\left(x + \int_0^t c \, dt\right), \quad (57)$$

---

\* In Appendix 2 an alternative point of view is presented (but not worked out) in which the sudden increase in propagation velocity does not occur.

$$\epsilon = \frac{\partial s}{\partial x} = f' \left( x + \int_0^t c dt \right), \quad (58)$$

$$v = \frac{\partial s}{\partial t} = c f' \left( x + \int_0^t c dt \right) = c \epsilon. \quad (59)$$

At time  $t$  let the front extend from  $x = 0$  to  $x = L$ ; then the momentum of that part of the string which composes the front is

$$M = \int_0^L \rho A v dx = \rho A c \int_0^L \epsilon dx = \rho A c s_L \quad (60)$$

where  $A$  is the cross-section area of the string, and  $s_L$  is the displacement at the right hand boundary of the front. The stress at this boundary is

$$\sigma_L = \sigma(\epsilon_L) \quad (61)$$

where  $\epsilon_L$  is the strain at this boundary.

Applying the momentum theorem (37) to that region which contains the front, and which moves with the front, we obtain

$$\frac{d}{dt} (\rho A c s_L) + \rho A c (c \epsilon_L) = A \sigma_L$$

or, canceling  $A$  and noting that all quantities except  $c$  and  $t$  are constant,

$$\rho s_L \frac{dc}{dt} + \rho \epsilon_L c^2 = \sigma_L. \quad (62)$$

This differential equation can be solved exactly; however, it is evident by inspection that, regardless of its initial value,  $c$  is going to approach the value obtained by placing  $dc/dt = 0$  in this equation. This final value of  $c$  is

$$c = \sqrt{\frac{\sigma_L}{\rho \epsilon_L}} \quad (63)$$

regardless of the shape of the front.

We note that this value agrees with (47), which applies to an abrupt wave front.

Next, let us suppose that  $c$  has attained its final value (63). We shall consider the energy balance of the same region to which we applied the momentum theorem, namely that which contains the front and moves with the front. Since the region moves with the front, we see that the rate of increase of kinetic energy and the rate of increase of potential (strain) energy in this region are both zero. Also, noting (59) we see that

$$\text{External power input} = A \sigma_L c \epsilon_L, \quad (64)$$

$$\begin{aligned} \text{Kinetic energy loss per second} \\ \text{through convection} &= \frac{A \rho c}{2} (c \epsilon_L)^2 \\ &= \frac{1}{2} A \sigma_L c \epsilon_L \end{aligned} \quad (65)$$

due to (63). Also,

$$\begin{aligned} \text{potential (strain) energy lost per} \\ \text{second through convection} &= c A \int_0^{\epsilon_L} \sigma d\epsilon \\ &= A \sigma_{ave} c \epsilon_L. \end{aligned} \quad (66)$$

For there to be an energy balance it is necessary and sufficient that

$$\sigma_{ave} = \frac{1}{2} \sigma_L. \quad (67)$$

Here  $\sigma_{ave}$  is the average value of  $\sigma$ , averaged on  $\epsilon$  from 0 to  $\epsilon_L$ . We note that the condition (67) is independent of the shape of the front. In fact the above relations agree with those on pages 29 and 30, which pertain to an abrupt wave front. It follows that the situation in regard to energy balance cannot be improved or even altered by changing the shape of the front.

Overtaking of a Wave Front by an Infinitesimal Wave. On pages 31 to 35 it was shown that with a concave upward stress-strain curve one abrupt front overtaking another will not smoothly merge with it. To be certain that the situation doesn't appear different when the overtaking front is treated as a series of infinitesimals let us consider the case where an infinitesimal wave comprising a strain increase  $d\epsilon$  and moving with a velocity

$$c_1 > c \quad (68)$$

overtakes a wave front comprising a strain increase  $\epsilon$  and moving with a velocity  $c$ , as indicated in Fig. 14. From (38), (41), and equation (138) of the appendix we have

$$v = c\epsilon, \quad (69)$$

$$c = \sqrt{\sigma/\rho\epsilon}, \quad (70)$$

$$c_1 = \sqrt{\sigma'(\epsilon)/\rho}, \quad (71)$$

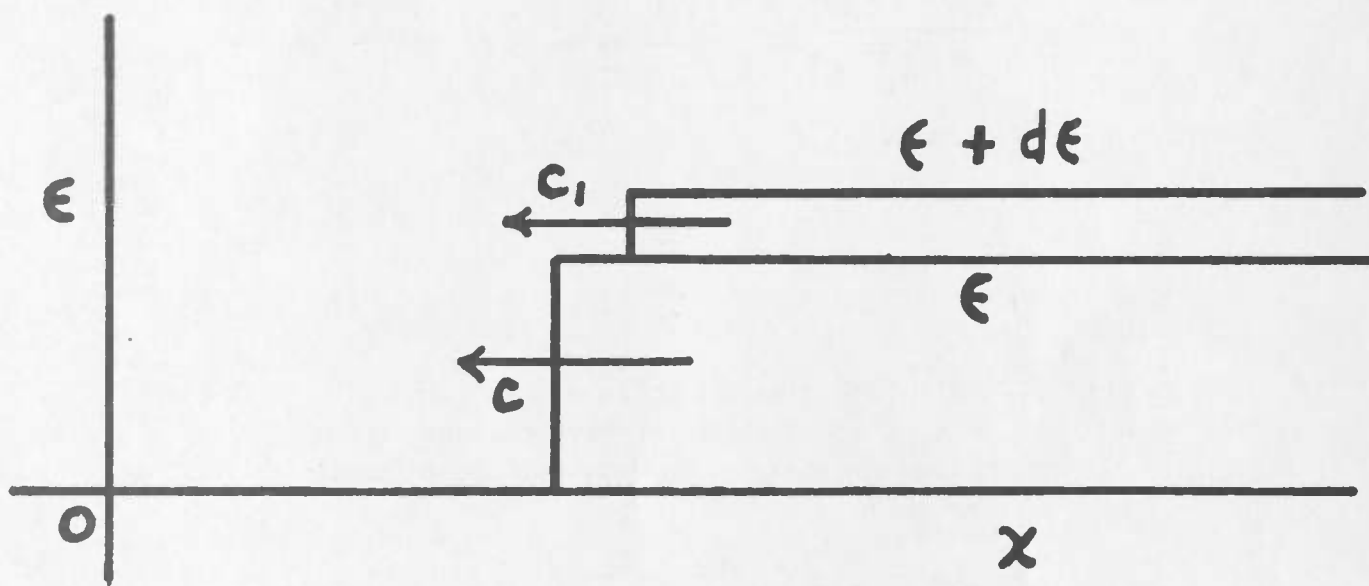


Fig. 14. Vertical wave front being overtaken by an infinitesimal wave.

where  $\sigma = \sigma(\epsilon)$ , and  $\sigma''(\epsilon) > 0$ . The stress-strain curve is thus concave upward, and  $c$  increases with  $\epsilon$ . Differentiating (69),  $\sigma$ , and (70), we obtain

$$dv = c d\epsilon + \epsilon dc, \quad (72)$$

$$d\sigma = \sigma'(\epsilon) d\epsilon, \quad (73)$$

$$dc = \frac{\epsilon d\sigma - \sigma d\epsilon}{2\sqrt{\frac{\rho\sigma}{\epsilon}} \epsilon^2} = \frac{\sigma'(\epsilon) - \frac{\sigma}{\epsilon}}{2\rho c \epsilon} d\epsilon,$$

or, noting (70) and (71),

$$dc = \frac{c_1^2 - c^2}{2c\epsilon} d\epsilon. \quad (74)$$

Equations (72) and (74) give the increases in  $v$  and  $c$ , respectively, when the infinitesimal wave catches up with the main wave, thereby increasing it by  $d\epsilon$ . Substituting (74) in (72) now gives

$$\begin{aligned} dv &= c d\epsilon + \frac{c_1^2 - c^2}{2c} d\epsilon \\ &= \frac{c_1^2 + c^2}{2c} d\epsilon. \end{aligned} \quad (75)$$

Since  $c_1 > c$ , which follows from (70), (71), and the fact that the stress-strain curve is concave upward, we see from (74) and (75) that  $dc$  and  $dv$  are both positive. These results are in agreement with those obtained on pages 32 and 35.

Consideration of a Discrete System of Masses and Springs as an Approximation to a Continuous System. We have shown (see pages 35-38 ) that no wave front of any constant shape can exist when the stress-strain curve is concave upward. In order to get some idea of what is going on, without becoming involved with a lot of calculations, we shall consider the discrete system of masses and springs shown in Fig. 15. The masses are represented by the numbered circles, are all equal, and extend indefinitely to the left. The nonlinear springs which connect adjacent masses are all alike, and are represented as sagging cords. Until the slack is taken up these cords carry no tension whatsoever; but as soon as the slack is taken up they behave as infinitely stiff elastic springs. The corresponding stress-strain curve is shown in Fig. 16, though here the role of  $\sigma$  is played by tension, and the role of  $\epsilon$  is played by the loss of slack, or relative displacement. The masses are all initially at rest as shown; and starting at time  $t = 0$  the right end moves to the right with a constant velocity  $v_0$ . We note that the stress-strain curve, though consisting of a broken line, lies in the category of concave upward.

Whenever the slack is taken up in any of the cords, impact conditions prevail. If the cord is that to the right of mass 1, this mass changes its velocity abruptly from 0 to  $2v_0$ . If the cord lies between two masses, those masses abruptly exchange their velocities. These facts are enough to enable one to draw the displacement-time curves for the various masses, as shown in Fig. 17. In drawing these curves we work from left to right, and note that slope is velocity. The slack is four divisions; and when the vertical distance between adjacent curves attains this value, impact occurs and slopes change abruptly.

As time goes on, the number of masses involved in the motion increases; also, since slope is velocity we see that of these masses alternate ones are at rest, and the other alternate ones are moving with a velocity  $2v_0$ , this being true at any time. The velocity distribution along the masses may hence be regarded as composed of a constant velocity  $v_0$  plus one which alternates between  $v_0$  and  $-v_0$  as we pass from mass to mass. The alternating velocity does not contribute to the overall motion, but it doubles



Fig. 15. String composed of discrete masses connected by slack cords.

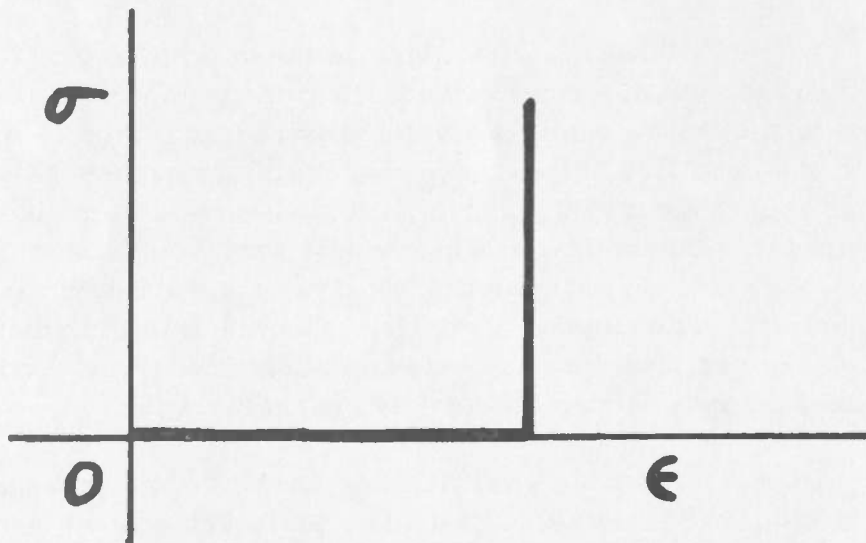


Fig. 16. Stress-strain curve of string in Fig. 15.

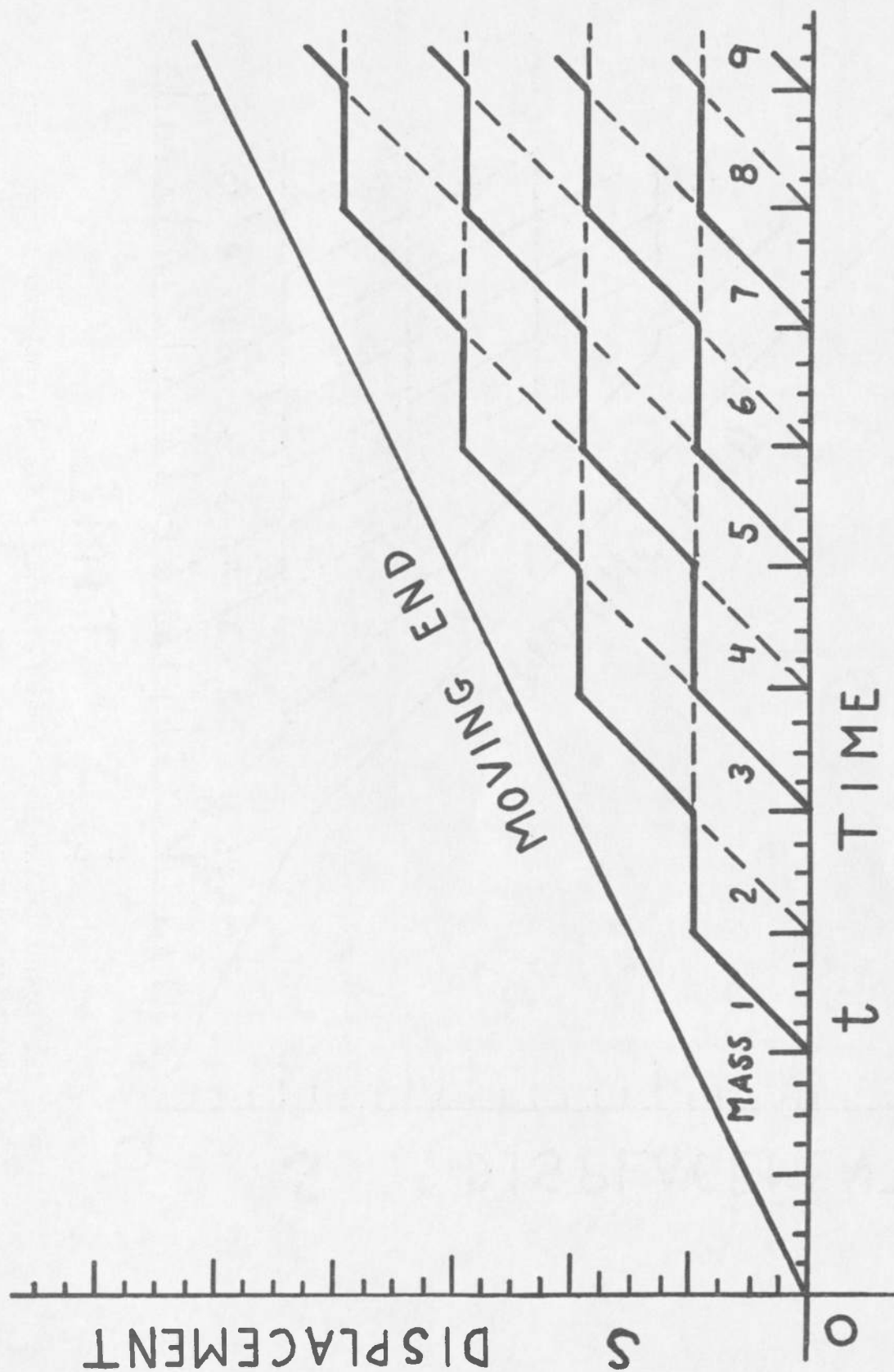


Fig. 17. Displacement-time curve for string of Fig. 15 subjected to constant-velocity impact.

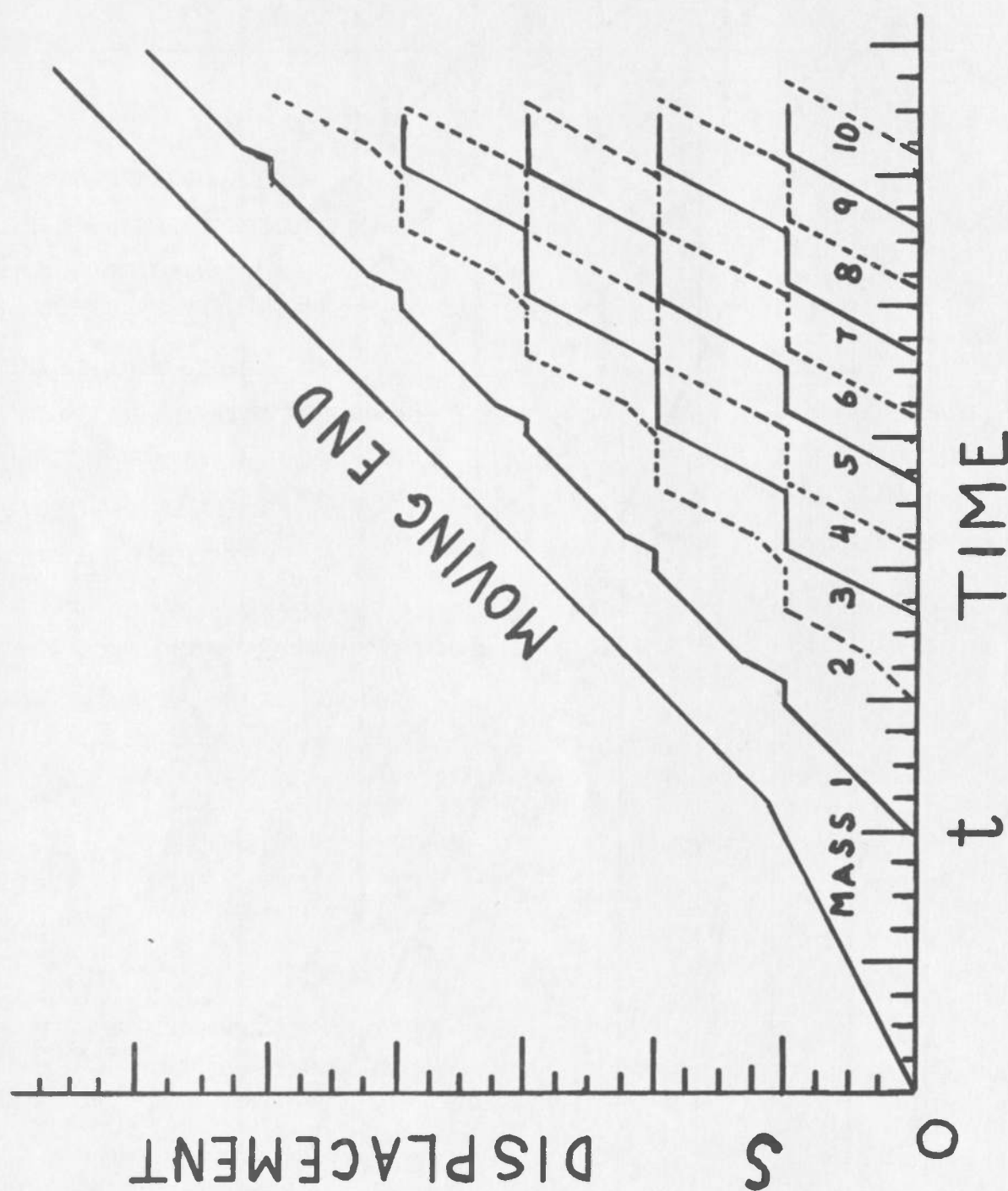


Fig. 18. Displacement-time curve for string of Fig. 15 subjected to an impact that changes in velocity.

the kinetic energy contained in those masses which are involved in the motion. We see that although the disturbance moves to the left at a uniform rate, it leaves behind it not a series of masses all moving with the same velocity  $v_0$ , but a motion having in addition a violent oscillation of amplitude  $v_0$ . We have thus encountered a situation similar to that which we have been led to believe takes place in a string whose stress-strain curve is concave upward, after a shock wave has formed.

Figure 18 is similar to Fig. 17, but corresponds to a case wherein the velocity of the moving end is altered during the course of the motion. It was thought that possibly such an alteration would cause a permanent scrambling of the pattern of motion; but such is evidently not the case.

In an attempt to approach the continuous distribution let us now divide each mass into  $n$  equal masses, and distribute these evenly along the string, the total mass per unit length remaining the same. The slack in the string between adjacent masses is thereby also reduced in the ratio  $1/n$ . Proceeding as before, we again obtain a motion wherein alternate masses of those involved in the disturbance are at rest, and the remaining alternate masses are moving with a velocity  $2v_0$ . Impacts occur  $n$  times as frequently as before, and the disturbance progresses to the left at the same constant rate as before. In the disturbed region the velocity distribution may be considered to be composed of the constant velocity  $v_0$  plus a violent oscillation of amplitude  $v_0$  as before; however, the wave length is now  $1/n$  times what it was before. If, now,  $n$  is increased indefinitely in order to approach a continuous mass distribution, this wave length approaches zero; and the oscillation retains its violence, but becomes infinitely fine.

It may be mentioned that in the case of discrete masses even if the springs were linear an end disturbance would not be propagated without distortion; for we are dealing with the mechanical analogue of a low pass filter. Different frequencies are propagated with different velocities; and above a certain cutoff frequency are not propagated at all.

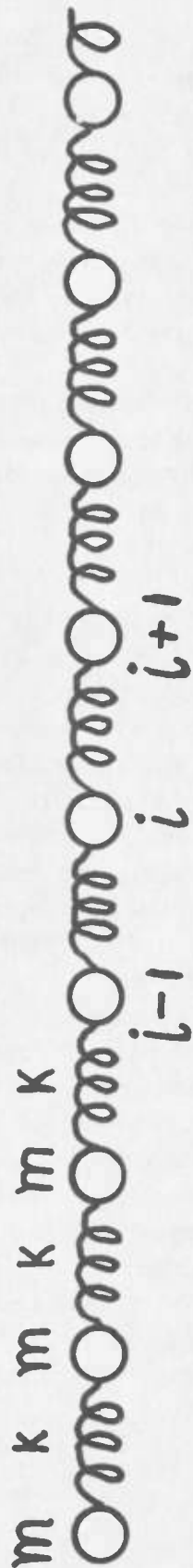


Fig. 19. String composed of discrete masses and linear springs.

More explicitly let us consider the linear system of Fig. 19. The masses are all of magnitude  $M$ , and the spring constants are all of value  $K$ . The masses are numbered from left to right, and the displacements toward the right are indicated by  $s_i$ . The equations of motion are

$$K(s_{i+1} - 2s_i + s_{i-1}) = M \frac{d^2 s_i}{dt^2}. \quad (76)$$

For vibratory motion we place

$$s_i = y_i \sin(\omega t + \gamma); \quad (77)$$

hence, (76) becomes

$$y_{i+1} - 2\left(1 - \frac{\omega^2}{\omega_0^2}\right)y_i + y_{i-1} = 0, \quad (78)$$

where

$$\omega_0 = \sqrt{\frac{2K}{M}}. \quad (79)$$

We note that  $\omega_0$  is the resonant angular frequency of any mass if the two adjacent masses are held fixed. If

$$\left|1 - \frac{\omega^2}{\omega_0^2}\right| < 1 \quad (80)$$

the solution of the difference equation (78) is

$$y_i = C_1 \cos i\alpha + C_2 \sin i\alpha \quad (81)$$

where

$$\alpha = \cos^{-1}\left(1 - \frac{\omega^2}{\omega_0^2}\right). \quad (82)$$

Equation (81) in (77) gives a standing wave of arbitrary amplitude, phase, and position; however, by superimposing two of these, thus

$$\begin{aligned} y_i &= c \sin i\alpha \cos \omega t - c \cos i\alpha \sin \omega t \\ &= c \sin(i\alpha - \omega t), \end{aligned} \quad (83)$$

we obtain a traveling wave. This wave moves to the right along the  $i$ -axis with a velocity

$$v = \frac{\omega}{\alpha}, \quad (84)$$

which, in view of (82), depends upon the angular frequency  $\omega$ . In this wave only integral values of  $i$  pertain to the masses, and thus have direct physical significance. Since  $\omega$  is positive, (80) is equivalent to

$$0 < \omega < \sqrt{2} \omega_0. \quad (85)$$

If

$$\omega > \sqrt{2} \omega_0 \quad (86)$$

(80) is violated; and in place of the traveling wave (83) we obtain the attenuated wave

$$y_i = c(-1)^i e^{-i\alpha} \sin(\omega t + \gamma) \quad (87)$$

where

$$\alpha = \cosh^{-1} \left( \frac{\omega^2}{\omega_0^2} - 1 \right). \quad (88)$$

Finally, if  $\omega = 0$  there is no motion. We now see that a wave is propagated only if  $\omega$  lies below the cutoff value  $\sqrt{2} \omega_0$ , in which case the velocity of propagation depends on  $\omega$ .

Oscillations in the Continuous System. The oscillations which we have encountered in the discrete system just considered (see page 45) suggest a possible way out of the difficulty which we found in connection with the lack of energy balance in the propagation of an abrupt, or square wave (see page 31). It may be that the extra energy which is available in the case where the stress-strain curve is concave upward is used to produce oscillations in that part of the string which has been passed over by the wave front. In investigating this possibility we shall consider the string in the unstretched state as extending from  $x = -\infty$  to  $x = 0$ ; and at time  $t = 0$  the right end of the string is suddenly given a constant velocity  $v_0$  to the right.

We shall consider a resulting strain wave of the form

$$\begin{aligned} \epsilon = \epsilon_0 + a_1 \sin \Omega(x - c_0 t) + a_2 \cos \Omega(x - c_0 t) \\ + a_3 \sin \Omega(x + c_0 t) + a_4 \cos \Omega(x + c_0 t) \end{aligned} \quad (89)$$

where  $\epsilon_0$ ,  $c_0$ ,  $\Omega$ , and the  $a$ 's are constants, and where

$$\Omega = \frac{2\pi}{\lambda}, \quad (90)$$

$\lambda$  being the wave length. In addition to the constant value  $\epsilon_0$  we are thus providing for the possibility of two sinusoidal waves of wave length  $\lambda$  moving in opposite directions with velocity  $c_0$ . If these two waves were infinitesimal, we know that  $c_0$  would be the velocity corresponding to a strain  $\epsilon_0$  (see appendix 1, page 62). In any case we should expect  $c_0$  to approximate this velocity, which is  $\sqrt{\sigma'(\epsilon_0)/\rho}$ . We shall try to determine the various constants in (89) so as to satisfy various dynamical conditions, one of which is that the energy unbalance which we encountered on page 31 be removed.

The differential equation which determines the motion is

$$\rho \frac{\partial^2 \lambda}{\partial t^2} = \frac{\partial \sigma}{\partial \chi}, \quad (91)$$

or

$$\rho \frac{\partial^2 \lambda}{\partial t^2} = \sigma'(\epsilon) \frac{\partial \epsilon}{\partial \chi}. \quad (92)$$

At  $x = 0$  we have

$$\lambda = v_0 t \quad \text{and} \quad \frac{\partial^2 \lambda}{\partial t^2} = 0 \quad \text{when} \quad \chi = 0; \quad (93)$$

hence (92) gives

$$\frac{\partial \epsilon}{\partial \chi} = 0 \quad \text{when} \quad \chi = 0. \quad (94)$$

Substituting (89) in (94), we obtain

$$\Omega [a_1 \cos \Omega c_0 t + a_2 \sin \Omega c_0 t + a_3 \cos \Omega c_0 t - a_4 \sin \Omega c_0 t] = 0.$$

This relation will be satisfied for all values of  $t$  if and only if

$$a_3 = -a_1, \quad \text{and} \quad a_4 = a_2, \quad (95)$$

due to which (89) becomes

$$\begin{aligned} \epsilon = \epsilon_0 + a_1 [\sin \Omega (\chi - c_0 t) - \sin \Omega (\chi + c_0 t)] \\ + a_2 [\cos \Omega (\chi - c_0 t) + \cos \Omega (\chi + c_0 t)]. \end{aligned} \quad (96)$$

This may be written

$$\begin{aligned}\epsilon &= \epsilon_0 - 2a_1 \cos \Omega x \sin \Omega c_0 t + 2a_2 \cos \Omega x \cos \Omega c_0 t \\ &= \epsilon_0 + 2 \cos \Omega x (-a_1 \sin \Omega c_0 t + a_2 \cos \Omega c_0 t),\end{aligned}\quad (97)$$

or

$$\epsilon = \epsilon_0 + \epsilon_1 \cos \Omega x \sin (\Omega c_0 t + \gamma) \quad (98)$$

where  $\epsilon_1$  and  $\gamma$  are constants. We see that  $\epsilon$  is composed of a constant and a standing wave of wave length  $\lambda = 2\pi/\Omega$  and angular frequency  $\omega = \Omega c_0$ .

Integrating (98) with respect to  $x$  holding  $t$  constant, we obtain

$$\Delta = \epsilon_0 x + \frac{\epsilon_1}{\Omega} \sin \Omega x \sin (\Omega c_0 t + \gamma) + \psi(t) \quad (99)$$

where  $\psi(t)$  is an unknown function of  $t$ . But

$$s = v_0 t \text{ when } x = 0, \quad (100)$$

which with (99) gives

$$\psi(t) = v_0 t;$$

hence

$$\Delta = \epsilon_0 x + \frac{\epsilon_1}{\Omega} \sin \Omega x \sin (\Omega c_0 t + \gamma) + v_0 t. \quad (101)$$

Differentiating partially with respect to  $t$ , we obtain the following expression for the velocity

$$v = v_0 + \epsilon_1 c_0 \sin \Omega x \cos (\Omega c_0 t + \gamma). \quad (102)$$

The kinetic energy per unit length of string is

$$\begin{aligned}\frac{\rho A v^2}{2} &= \frac{\rho A}{2} \left[ v_0^2 + 2v_0 \epsilon_1 c_0 \sin \Omega x \cos (\Omega c_0 t + \gamma) \right. \\ &\quad \left. + \epsilon_1^2 c_0^2 \sin^2 \Omega x \cos^2 (\Omega c_0 t + \gamma) \right]\end{aligned}\quad (103)$$

where  $A$  is the cross-section area.

The average of this expression taken over a wave length  $\lambda = 2\pi/\Omega$  is

$$\begin{aligned} T_{\lambda} &= \frac{\Omega}{2\pi} \int_{x_0}^{x_0 + \frac{2\pi}{\Omega}} \frac{\rho A v^2}{2} dx \\ &= \frac{\rho A}{2} \left[ v_0^2 + \frac{\epsilon_1^2 c_0^2}{2} \cos^2(\Omega c_0 t + \delta) \right]; \end{aligned} \quad (104)$$

the average taken over a time period  $2\pi/\Omega c_0$  is

$$\begin{aligned} T_P &= \frac{\Omega c_0}{2\pi} \int_{t_0}^{t_0 + \frac{2\pi}{\Omega c_0}} \frac{\rho A v^2}{2} dt \\ &= \frac{\rho A}{2} \left[ v_0^2 + \frac{\epsilon_1^2 c_0^2}{2} \sin^2 \Omega x \right]; \end{aligned} \quad (105)$$

and the average over both is

$$\begin{aligned} T_{\lambda P} &= \frac{\Omega^2 c_0}{4\pi^2} \int_{x_0}^{x_0 + \frac{2\pi}{\Omega}} \int_{t_0}^{t_0 + \frac{2\pi}{\Omega c_0}} \frac{\rho A v^2}{2} dt dx \\ &= \frac{\rho A}{2} \left[ v_0^2 + \frac{\epsilon_1^2 c_0^2}{4} \right]. \end{aligned} \quad (106)$$

Here  $x_0$  and  $t_0$  are arbitrary values of  $x$  and  $t$ , respectively.

The strain energy, or potential energy, per unit length of string is

$$\Phi = A \int_0^{\epsilon} \sigma(\epsilon) d\epsilon \quad (107)$$

hence, before this can be averaged over a space period  $\lambda$  or a time period  $2\pi/\Omega c_0$ ,  $\sigma(\epsilon)$  must be specified. For the present let us place

$$\sigma(\epsilon) = E_0 \epsilon + E_1 \epsilon^2 \quad (108)$$

where  $E_0$  and  $E_1$  are both positive constants. We note that this stress-strain curve is concave upward. Equation (108) in (107) now gives

$$\Phi = \frac{AE_0\epsilon^2}{2} + \frac{AE_1\epsilon^3}{3}. \quad (109)$$

Substituting (98) in this expression, we obtain

$$\begin{aligned} \Phi = & \frac{AE_0}{2} [\epsilon_0^2 + 2\epsilon_0\epsilon_1 \cos \Omega\chi \sin(\Omega c_0 t + \gamma) \\ & + \epsilon_1^2 \cos^2 \Omega\chi \sin^2(\Omega c_0 t + \gamma)] \\ & + \frac{AE_1}{3} [\epsilon_0^3 + 3\epsilon_0^2\epsilon_1 \cos \Omega\chi \sin(\Omega c_0 t + \gamma) \\ & + 3\epsilon_0\epsilon_1^2 \cos^2 \Omega\chi \sin^2(\Omega c_0 t + \gamma) \\ & + \epsilon_1^3 \cos^3 \Omega\chi \sin^3(\Omega c_0 t + \gamma)]. \end{aligned} \quad (110)$$

The average of this expression over a wave length is

$$\begin{aligned} \bar{\Phi}_\lambda = & \frac{\Omega}{2\pi} \int_{x_0}^{x_0 + \frac{2\pi}{\Omega}} \Phi dx \\ = & \frac{AE_0}{2} \left[ \epsilon_0^2 + \frac{\epsilon_1^2}{2} \sin^2(\Omega c_0 t + \gamma) \right] \\ & + \frac{AE_1}{3} \left[ \epsilon_0^3 + \frac{3}{2} \epsilon_0 \epsilon_1^2 \sin^2(\Omega c_0 t + \gamma) \right] \\ = & \frac{1}{6} AE_0^2 (3E_0 + 2E_1\epsilon_0) \\ & + \frac{1}{4} AE_1^2 (E_0 + 2E_1\epsilon_0) \sin^2(\Omega c_0 t + \gamma); \end{aligned} \quad (111)$$

the average over the time period  $2\pi/\Omega c_0$  is

$$\begin{aligned} \bar{\Phi}_p = & \frac{\Omega c_0}{2\pi} \int_{t_0}^{t_0 + \frac{2\pi}{\Omega c_0}} \Phi dt \\ = & \frac{1}{2} AE_0 \left[ \epsilon_0^2 + \frac{1}{2} \epsilon_1^2 \cos^2 \Omega\chi \right] \\ & + \frac{1}{3} AE_1 \left[ \epsilon_0^3 + \frac{3}{2} \epsilon_0 \epsilon_1^2 \cos^2 \Omega\chi \right] \\ = & \frac{1}{6} AE_0^2 (3E_0 + 2E_1\epsilon_0) \\ & + \frac{1}{4} AE_1^2 (E_0 + 2E_1\epsilon_0) \cos^2 \Omega\chi \end{aligned} \quad (112)$$

and the average over both is

$$\begin{aligned}\bar{\Phi}_{\lambda P} &= \frac{\Omega^2 \kappa_0}{4\pi^2} \int_{x_0}^{x_0 + \frac{2\pi}{\Omega}} \int_{x_0}^{x_0 + \frac{2\pi}{\Omega}} \bar{\Phi} dt d\lambda \\ &= \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) \\ &\quad + \frac{1}{8} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0). \quad (113)\end{aligned}$$

Adding (104) and (111) we obtain the following expression for the average of the total energy taken over a wave length.

$$\begin{aligned}T_\lambda + \bar{\Phi}_\lambda &= \frac{1}{2} \rho A \left( v_0^2 + \frac{\epsilon_1^2 \kappa_0^2}{4} \right) + \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) \\ &\quad + \frac{1}{8} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0) + \left[ \frac{1}{8} \rho A \epsilon_1^2 \kappa_0^2 \right. \\ &\quad \left. - \frac{1}{8} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0) \right] \cos 2(\Omega \kappa_0 t + \gamma). \quad (114)\end{aligned}$$

There is no mechanism for transferring energy to and from an appreciable length of the string at a frequency  $\Omega \kappa_0 \pi$ , which is twice the frequency of vibration of the standing wave. In order to avoid a situation where this is implied, we equate the coefficient of the cosine in (114) to zero, thus

$$\rho A \epsilon_1^2 \kappa_0^2 - A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0) = 0 \quad (115)$$

or

$$\kappa_0 = \sqrt{\frac{E_0 + 2E_1 \epsilon_0}{\rho}}. \quad (116)$$

This expression determines  $c_0$ . Noting (108) we see that (116) may be written

$$c_0 = \sqrt{\sigma'(\epsilon_0) / \rho}, \quad (117)$$

which is the wave velocity corresponding to a strain  $\epsilon_0$ . This is in line with the conjecture made on page 49. In view of (115), (114) reduces to

$$\begin{aligned} T_\lambda + \Phi_\lambda = & \frac{1}{2} \rho A (v_0^2 + \frac{\epsilon_1^2 c_0^2}{4}) \\ & + \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) + \frac{1}{8} \epsilon_1^2 (E_0 + 2E_1 \epsilon_0), \end{aligned} \quad (118)$$

or, substituting (116) for  $c_0$ ,

$$\left. \begin{aligned} T_\lambda + \Phi_\lambda = & \frac{1}{2} \rho A v_0^2 + \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) \\ & + \frac{1}{4} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0). \end{aligned} \right\} \quad (119)$$

Adding (105) and (112) we obtain the following expression for the average of the total energy per unit length taken over a time period  $2\pi / \Omega c_0$ .

$$\begin{aligned} T_p + \Phi_p = & \frac{1}{2} \rho A (v_0^2 + \frac{\epsilon_1^2 c_0^2}{4}) + \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) \\ & + \frac{1}{8} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0) - \left[ \frac{1}{8} \rho A \epsilon_1^2 c_0^2 \right. \\ & \left. - \frac{1}{8} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0) \right] \cos 2\Omega x. \end{aligned} \quad (120)$$

But here the coefficient of the cosine is zero due to (115); hence, (120) duplicates (118), or (119). We thus have finally, noting (106) and (113)

$$\begin{aligned} T_p + \Phi_p = T_\lambda + \Phi_\lambda = T_{\lambda p} + \Phi_{\lambda p} \\ = \frac{1}{2} \rho A v_0^2 + \frac{1}{6} A \epsilon_0^2 (3E_0 + 2E_1 \epsilon_0) + \frac{1}{4} A \epsilon_1^2 (E_0 + 2E_1 \epsilon_0). \end{aligned} \quad (121)$$

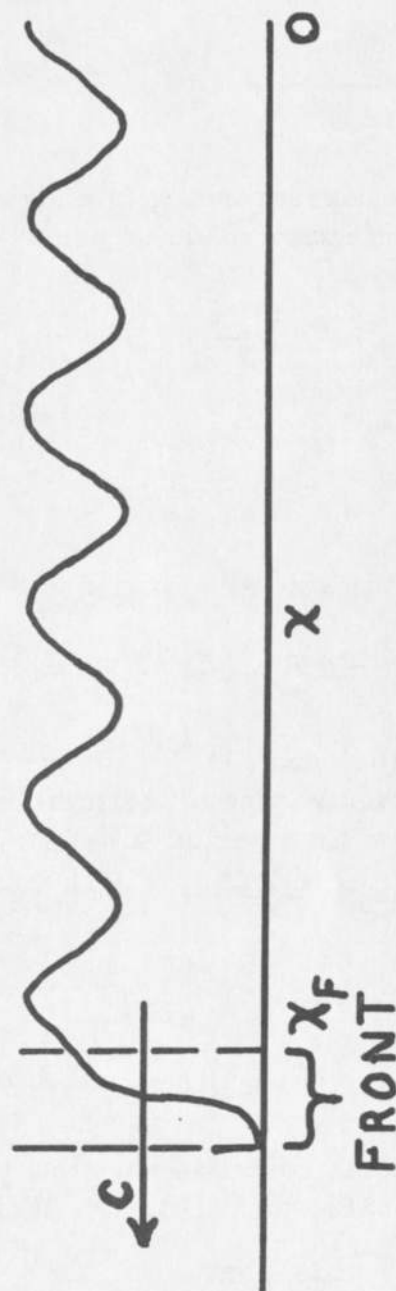


Fig. 20. Wave-front as a transition region between the unstretched and stretched string.

Next, let us suppose that the disturbance moves to the left with an average velocity  $\bar{c}$ . We do not require that there be an abrupt wave front, nor that the wave velocity  $c$  be constant. In fact at this point we do not have to know the detailed nature of this front - only that there is a transition region of finite length between the unstressed string and the string in the stressed state described above (see eq 98), which region moves to the left with an average velocity  $\bar{c}$ . We shall now proceed to obtain three conditions which must be satisfied for the conservation of mass, momentum, and energy.

Conservation of Mass. The total stretch which occurs in the transition region, or "front", will be denoted by  $s_F$ . We do not know this quantity; however, we do know that it is bounded. Referring to Fig. 20, let  $x_F$  be the abscissa of the right hand boundary of this front; then (101) gives

$$s_F = s(\chi_F) = \epsilon_0 \chi_F + \frac{\epsilon_1}{\Omega} \sin \Omega \chi_F \sin(\Omega c_0 t + \gamma) + v_0 t. \quad (122)$$

But

$$\chi_F = -\bar{c} t + \chi_1(t) \quad (123)$$

where  $\chi_1(t)$  is bounded. This in (122) gives

$$s_F = -\epsilon_0 \bar{c} t + \epsilon_0 \chi_1(t) + \frac{\epsilon_1}{\Omega} \sin \Omega \chi_F \sin(\Omega c_0 t + \gamma) + v_0 t. \quad (124)$$

As time goes on, all terms in this equation are bounded except the first and last terms on the right hand side. These terms must cancel each other; otherwise the equation will ultimately be violated. It follows that

$$-\epsilon_0 \bar{c} t + v_0 t = 0,$$

$$\bar{c} = \frac{v_0}{\epsilon_0} \quad (125)$$

Conservation of Momentum. The total momentum in the string is

$$M = M_F + A \rho \int_{x_F}^0 v dx, \quad (126)$$

where  $M_F$  is the momentum in the front, and is known only to be bounded. Substituting (102) in this expression we obtain

$$\begin{aligned} M &= M_F + A \rho \int_{x_F}^0 [v_0 + \epsilon_1 c_0 \sin \Omega x \cos (\Omega c_0 t + \gamma)] dx \\ &= M_F + A \rho \left[ -v_0 x_F + \frac{\epsilon_1 c_0}{\Omega} (\cos \Omega x_F - 1) \cos (\Omega c_0 t + \gamma) \right] \end{aligned} \quad (127)$$

This, however, must be equal to the total impulse applied at the right hand end; hence

$$M = A \int_0^t \sigma(\epsilon_{x=0}) dt. \quad (128)$$

Placing  $x = 0$  in (98), then substituting in (108) and (128), and then adding an unknown but bounded term  $M_{OF}$  to allow for the fact that a short time may be required for  $\epsilon$  to attain the form (98), we obtain

$$\begin{aligned} M &= A \int_0^t \left\{ E_0 [\epsilon_0 + \epsilon_1 \sin (\Omega c_0 t + \gamma)] + E_1 [\epsilon_0^2 + 2\epsilon_0 \epsilon_1 \sin (\Omega c_0 t + \gamma) + \epsilon_1^2 \sin^2 (\Omega c_0 t + \gamma)] \right\} dt + M_{OF} \\ &= A \left[ E_0 \epsilon_0 + E_1 \left( \epsilon_0^2 + \frac{\epsilon_1^2}{2} \right) \right] t - \frac{A(E_0 \epsilon_1 + 2E_1 \epsilon_0 \epsilon_1)}{\Omega c_0} [\cos (\Omega c_0 t + \gamma) - \cos \gamma] \\ &\quad - \frac{A E_1 \epsilon_1^2}{4 \Omega c_0} [\sin 2(\Omega c_0 t + \gamma) - \sin 2\gamma] + M_{OF}. \end{aligned} \quad (129)$$

Replacing  $x_F$  in (127) by (123), then equating the unbounded terms in (127) and (129), we obtain

$$\begin{aligned} A \rho v_0 \bar{\epsilon} &= A \left[ E_0 \epsilon_0 + E_1 \left( \epsilon_0^2 + \frac{\epsilon_1^2}{2} \right) \right] \\ \rho v_0 \bar{\epsilon} &= E_0 \epsilon_0 + E_1 \left( \epsilon_0^2 + \frac{\epsilon_1^2}{2} \right). \end{aligned} \quad (130)$$

Conservation of Energy. The energy imparted to the string by the force at the right end is

$$\mathcal{E} = A v_0 \int_0^t \sigma(\epsilon_{x=r_0}) dt. \quad (131)$$

This expression differs from (128) only by a factor of  $v_0$ , and can hence be obtained by multiplying (129) by  $v_0$ , thus

$$\begin{aligned} \mathcal{E} &= A v_0 \left[ E_0 \epsilon_0 + E_1 \left( \epsilon_0^2 + \frac{\epsilon_1^2}{2} \right) \right] t \\ &\quad - \frac{A v_0}{\Omega c_0} (E_0 \epsilon_1 + 2 E_1 \epsilon_0 \epsilon_1) [\cos(\Omega c_0 t + \gamma) - \cos \gamma] \\ &\quad - \frac{A v_0 E_1 \epsilon_1^2}{4 \Omega c_0} [\sin 2(\Omega c_0 t + \gamma) - \sin 2\gamma] + v_0 M_{OF}. \end{aligned} \quad (132)$$

The total energy, kinetic and potential, contained in the string is

$$T + \Phi = (T_\lambda + \Phi_\lambda) \bar{\epsilon} t + \mathcal{E}_B \quad (133)$$

where  $\mathcal{E}_B$  is bounded. Substituting (119) in (133), then equating the unbounded terms of (132) and (133), we obtain

$$\begin{aligned} \left[ \frac{\rho A v_0^2}{2} + \frac{1}{6} A \epsilon_0^2 (3 E_0 + 2 E_1 \epsilon_0) \right. \\ \left. + \frac{1}{4} A \epsilon_1^2 (E_0 + 2 E_1 \epsilon_0) \right] \bar{\epsilon} &= A v_0 \left[ E_0 \epsilon_0 \right. \\ &\quad \left. + E_1 \left( \epsilon_0^2 + \frac{\epsilon_1^2}{2} \right) \right], \end{aligned}$$

or noting (130)

$$\frac{1}{6}\epsilon_0^2(3E_0+2E_1\epsilon_0) + \frac{1}{4}\epsilon_1^2(E_0+2E_1\epsilon_0) = \frac{1}{2}\rho v_0^2,$$

$$2\epsilon_0^2(3E_0+2E_1\epsilon_0) + 3\epsilon_1^2(E_0+2E_1\epsilon_0) = 6\rho v_0^2. \quad (134)$$

Substituting (125) in (130), we have

$$4\epsilon_0^2(E_0+E_1\epsilon_0) + \epsilon_1^2 E_1\epsilon_0 = 2\rho v_0^2, \quad (135)$$

or

$$\epsilon_1^2 = \frac{2}{E_1\epsilon_0} \left[ \rho v_0^2 - \epsilon_0^2(E_0+E_1\epsilon_0) \right]. \quad (136)$$

Substituting (136) in (134), we obtain finally

$$E_1\epsilon_0^3(3E_0+2E_1\epsilon_0) + 3(E_0+2E_1\epsilon_0) \cdot \left[ \rho v_0^2 - \epsilon_0^2(E_0+E_1\epsilon_0) \right] = 3\rho v_0^2 E_1\epsilon_0,$$

or

$$4E_1^2\epsilon_0^4 + 6E_1E_0\epsilon_0^3 + 3E_0^2\epsilon_0^2 - 3\rho v_0^2 E_1\epsilon_0 - 3\rho v_0^2 E_0 = 0. \quad (137)$$

This quartic equation determines  $\epsilon_0$ , after which  $\epsilon_1$  is given by (136), and  $\bar{\epsilon}$  by (125).

Thus we have shown that for a particular concave-upward stress-strain curve as given by (108) it is possible to specify the constants in an oscillatory solution (98) to a tensile impact problem so as to give an energy balance. The implication is that for any concave-upward stress-strain curve the solution would be

oscillatory. However, since we have seen that the iterative procedure used in our computer calculations will produce oscillations at a discontinuity even when they are not required for energy balance it would be pointless to try to account for the oscillations observed in the computer results from energy considerations alone.

Before leaving this theoretical analysis of oscillations it should be pointed out that in some instances problems have been left without working out all the details and without utilizing all the known conditions of the problem. This was done because we were anxious to get on with solutions involving creep of the material. We believe that creep is always present and that a time-independent model can never accurately represent reality. Hence, although a more complete study of the time-independent model could be made, and although such a study might help in the understanding of the propagation of actual strain waves, we will not develop the time-independent analysis further at this time.

## 8. Conclusion

It has been shown in the present report that a non-linear time-independent tensile impact problem may be solved by an iterative computer procedure and that the result agrees with the solution obtained by the method-of-characteristics. In either method difficulty may arise if the stress-strain curve becomes concave upward. In the method-of-characteristics these difficulties are overcome by assuming shock conditions (and by implication at least assuming the production of heat). In the iterative computer procedure the difficulties cause oscillations. Although partly computational it has been shown that in many ways these oscillations are a more correct solution than the assumed shock.

We are now ready to apply the iterative computer procedure to the time dependent non-linear case which we believe is closer to reality than anything discussed so far. This will be done in Report III.

### Appendix 1

Independence of Infinitesimal Strain Waves. The purpose of this section is to check the validity of the idea that in the case where  $\sigma = \sigma(\epsilon)$  each infinitesimal strain increment travels with its own velocity

$$c(\epsilon) = \sqrt{\sigma'(\epsilon)/\rho} \quad (138)$$

independent of the others. This idea is suggested by the differential equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \quad (139)$$

written in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial \epsilon}{\partial x} \quad (140)$$

$$= c^2 \frac{\partial^2 u}{\partial x^2} \quad (141)$$

where  $c$  is given by (138). For an infinitesimal disturbance,  $c$  is essentially constant, (141) becomes the wave equation closely, and it appears that the disturbance is propagated undistorted at velocity  $c(\epsilon)$ . If the disturbance is an infinitesimal strain increment  $d\epsilon$ , we have the above-mentioned idea. Although the above provides a suggestion, it leaves a great deal to be desired by way of proof. In what follows we shall suppose that the idea is valid, and shall determine whether or not the strain wave which it gives satisfies the differential equation.

At time  $t$  let the strain wave be represented by the curve in Fig. 21. As time goes on, the future shape of this curve is determined by the motion of the individual strain increments — each shearing by the others at its own velocity  $c(\epsilon)$ , as indicated. In time  $dt$  each strain increment  $d\epsilon$  moves to the left a distance  $c dt$ . It follows that at abscissa  $x$  (of the unstretched string) the strain increases during this time by the amount

$$d\epsilon = \frac{\partial \epsilon}{\partial x} c(\epsilon) dt ;$$

hence, dividing by  $dt$ ,

$$\frac{\partial \epsilon}{\partial t} = c(\epsilon) \frac{\partial \epsilon}{\partial x} , \quad (142)$$

which is a quasilinear first order partial differential equation. Equation (142) does not resemble the desired equation (140) or (141); and cannot be transformed into it in any evident manner. However, there is still the possibility that the solution of (142) will satisfy (141). We shall therefore solve (142). Equation (142) may be written in the standard form

$$\frac{\partial \epsilon}{\partial t} - c(\epsilon) \frac{\partial \epsilon}{\partial x} = 0 , \quad (143)$$

from which we see that the characteristics are given by the equations

$$\frac{dt}{1} = \frac{dx}{-c(\epsilon)} = \frac{d\epsilon}{0} . \quad (144)$$

These give

$$\epsilon = c_2 = \text{constant}, \quad (145)$$

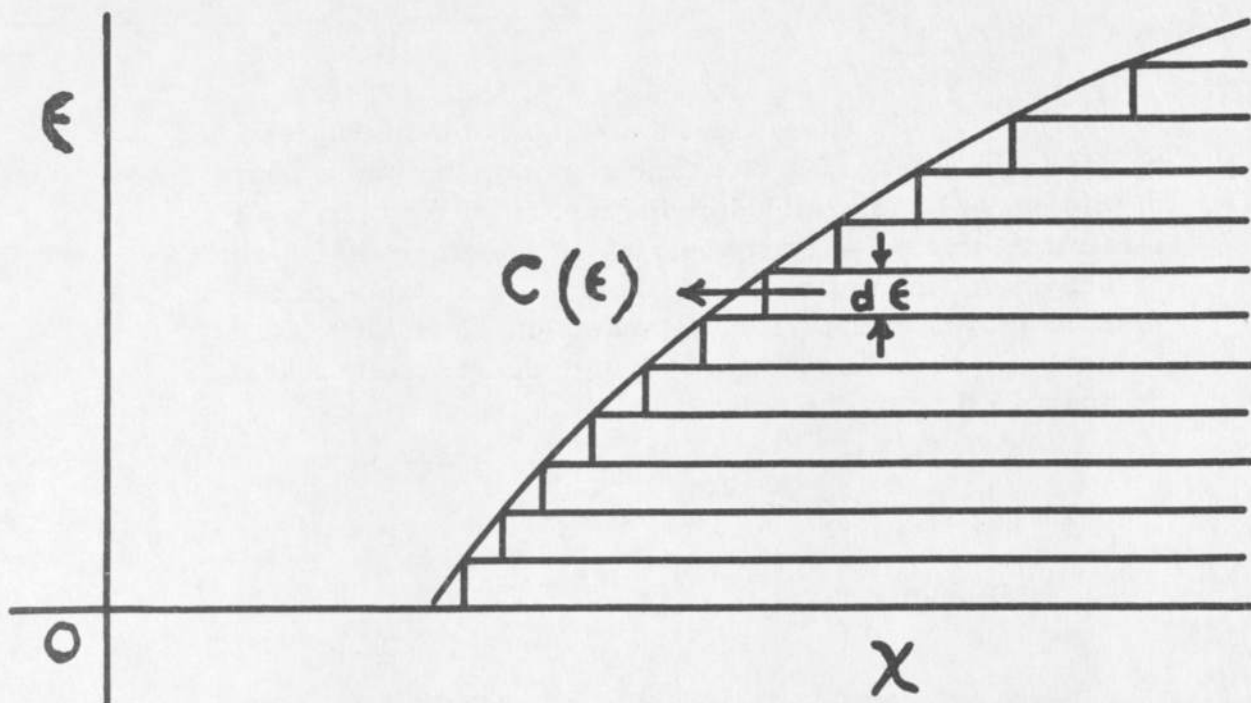


Fig. 21. Strain-wave front composed of incremental strains each moving at its own velocity.

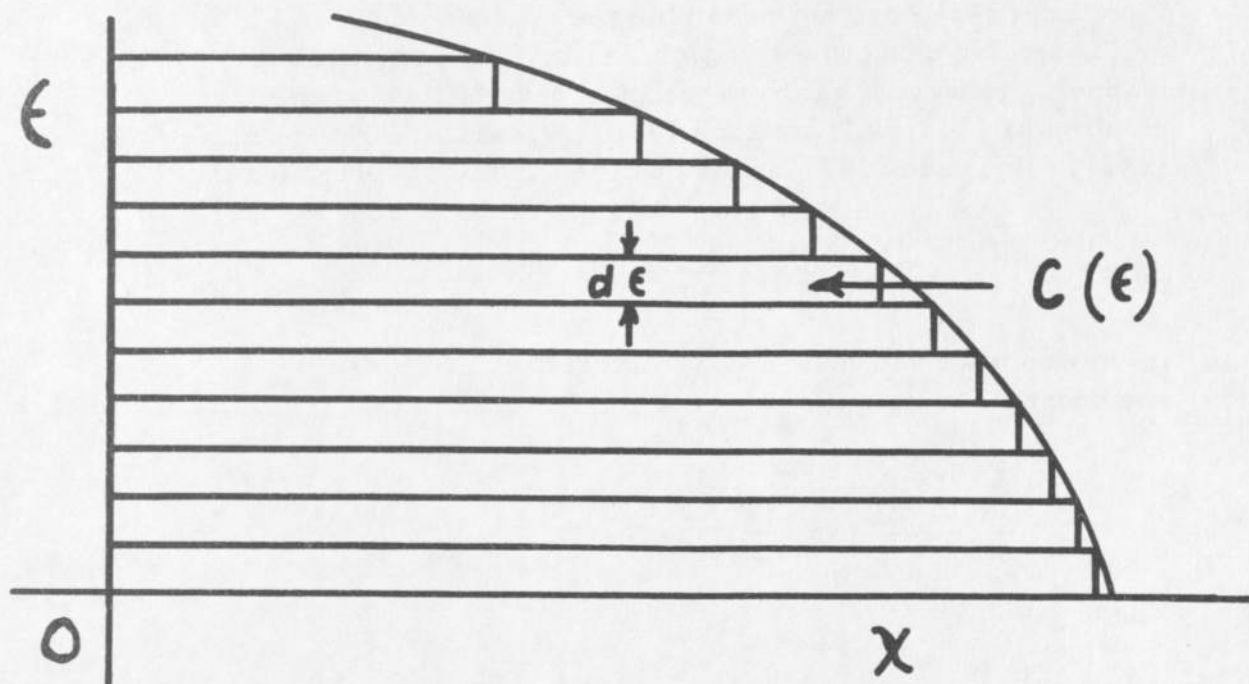


Fig. 22. Strain-wave front with negative slope composed of incremental waves.

whence

$$dx + c(c_2) dt = 0$$

$$x + c(c_2)t = c_1 = \text{constant}. \quad (146)$$

Replacing  $c_2$  by  $\epsilon$  we obtain for the general solution of (143), and hence (142)

$$\epsilon = f[x + c(\epsilon)t] \quad (147)$$

where  $f$  is an arbitrary function. Initially  $t = 0$  and (147) becomes

$$\epsilon = f(x) \quad (148)$$

from which we see that  $f(x)$  is the arbitrarily specified initial distribution of  $\epsilon$ .

From (147) we obtain

$$d\epsilon = (f')(dx + c'(\epsilon)t d\epsilon + c d\epsilon) \quad (149)$$

where

$$\left. \begin{aligned} f' &= f'[x + c(\epsilon)t] \\ c &= c(\epsilon) \\ c' &= c'(\epsilon). \end{aligned} \right\} \quad (150)$$

Equation (149) may be written

$$(1 - f'c't) d\epsilon = (f')(dx + c d\epsilon) \quad (151)$$

from which it follows that

$$\frac{\partial \epsilon}{\partial x} = \frac{f'}{1 - f'c't}, \quad (152)$$

$$\frac{\partial \epsilon}{\partial t} = \frac{c f'}{1 - f'c't}. \quad (153)$$

We note that these quantities satisfy the differential equation (142). In order to check the differential equation (141) we need  $\partial^2 s / \partial t^2$  which we shall now proceed to calculate.

Since

$$\epsilon = \frac{\partial u}{\partial x} \quad (154)$$

we obtain  $s$  by integrating  $\epsilon$  with respect to  $x$  holding  $t$  constant, thus

$$u = \int_0^x f[x + c(\epsilon)t] dx + u(0, t), \quad (155)$$

where  $s(0, t)$  is the value of  $s$  at  $x = 0$  and time  $t$ . It follows that

$$\left. \begin{aligned} \frac{\partial u}{\partial t} = \int_0^x f'[x + c(\epsilon)t] [c(\epsilon) + t c'(\epsilon) \frac{\partial \epsilon}{\partial t}] dx \\ + \frac{d}{dt} u(0, t). \end{aligned} \right\} \quad (156)$$

Since, noting (152) and (153)

$$\begin{aligned} \frac{\partial}{\partial x} [x + c(\epsilon)t] &= 1 + t c'(\epsilon) \frac{\partial \epsilon}{\partial x} \\ &= \frac{1}{c(\epsilon)} [c(\epsilon) + t c'(\epsilon) \frac{\partial \epsilon}{\partial x}], \end{aligned} \quad (157)$$

(156) may be written

$$\left. \begin{aligned} \frac{\partial u}{\partial t} = \int_0^x f'[x + c(\epsilon)t] c(\epsilon) \frac{\partial}{\partial x} [x + c(\epsilon)t] dx \\ + \frac{d}{dt} u(0, t). \end{aligned} \right\} \quad (158)$$

Integrating by parts, we obtain

$$\frac{\partial \mathcal{A}}{\partial t} = \left\{ f[x + \mathcal{L}(\epsilon)t] \mathcal{L}(\epsilon) \right\}_0^x - \int_0^x f[x + \mathcal{L}(\epsilon)t] \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial x} dx + \frac{d}{dt} \mathcal{A}(0, t). \quad (159)$$

Differentiating again, we have

$$\left. \begin{aligned} \frac{\partial^2 \mathcal{A}}{\partial t^2} = & \left\{ f[x + \mathcal{L}(\epsilon)t] \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial t} \right. \\ & + f'[x + \mathcal{L}(\epsilon)t] \mathcal{L}(\epsilon) [\mathcal{L}(\epsilon) + t \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial t}] \Big\}_0^x \\ & + \frac{d^2}{dt^2} \mathcal{A}(0, t) \\ & - \int_0^x \left\{ f'[x + \mathcal{L}(\epsilon)t] [\mathcal{L}(\epsilon) + t \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial t}] \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial x} \right. \\ & \left. + f[x + \mathcal{L}(\epsilon)t] \mathcal{L}''(\epsilon) \frac{\partial \epsilon}{\partial t} \frac{\partial \epsilon}{\partial x} + f[x + \mathcal{L}(\epsilon)t] \mathcal{L}'(\epsilon) \frac{\partial^2 \epsilon}{\partial t \partial x} \right\} dx. \end{aligned} \right\} \quad (160)$$

Here in the first bracket we note from (147) that

$$\frac{\partial \epsilon}{\partial t} = f'[x + \mathcal{L}(\epsilon)t] [\mathcal{L}(\epsilon) + t \mathcal{L}'(\epsilon) \frac{\partial \epsilon}{\partial t}] ; \quad (161)$$

also, since, noting (152) and (153),

$$\frac{\partial \epsilon}{\partial t} = \mathcal{L}(\epsilon) \frac{\partial \epsilon}{\partial x} \quad (162)$$

it follows that

$$\frac{\partial^2 \epsilon}{\partial t \partial x} = \mathcal{L}(\epsilon) \frac{\partial^2 \epsilon}{\partial x^2} + \mathcal{L}'(\epsilon) \left( \frac{\partial \epsilon}{\partial x} \right)^2. \quad (163)$$

Furthermore

$$\left. \begin{aligned} & \frac{\partial}{\partial x} \left\{ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right\} \\ &= f'[x+c(\epsilon)t] \left\{ 1 + t c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right\} c'(\epsilon) \frac{\partial \epsilon}{\partial x} \\ &+ f[x+c(\epsilon)t] c''(\epsilon) \left( \frac{\partial \epsilon}{\partial x} \right)^2 \\ &+ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial^2 \epsilon}{\partial x^2} . \end{aligned} \right\} \quad (164)$$

In view of these relations the integrand of the integral in (160) becomes

$$\left. \begin{aligned} \text{integrand} &= c(\epsilon) \frac{\partial}{\partial x} \left\{ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right\} \\ &+ f[x+c(\epsilon)t] \left\{ c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right\}^2 \end{aligned} \right\} \quad (165)$$

and (160) becomes

$$\left. \begin{aligned} \frac{\partial^2 \Lambda}{\partial t^2} &= \left\{ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} + c(\epsilon) \frac{\partial \epsilon}{\partial t} \right\}_0^x \\ &+ \frac{d^2}{dt^2} \Lambda(0, t) \\ &- \int_0^x \left\{ c(\epsilon) \frac{\partial}{\partial x} \left[ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right] \right. \\ &\quad \left. + f[x+c(\epsilon)t] \left[ c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right]^2 \right\} dx . \end{aligned} \right\} \quad (166)$$

Integrating by parts, we now obtain

$$\left. \begin{aligned} \frac{\partial^2 \Lambda}{\partial t^2} &= \left\{ f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} + c(\epsilon) \frac{\partial \epsilon}{\partial t} \right. \\ &\quad \left. - c(\epsilon) f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right\}_0^x \\ &+ \frac{d^2}{dt^2} \Lambda(0, t) \\ &+ \int_0^x \left\{ c'(\epsilon) \frac{\partial \epsilon}{\partial x} f[x+c(\epsilon)t] c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right. \\ &\quad \left. - f[x+c(\epsilon)t] \left[ c'(\epsilon) \frac{\partial \epsilon}{\partial x} \right]^2 \right\} dx . \end{aligned} \right\} \quad (167)$$

Noting (162) we see that the first and last terms in the first bracket cancel each other; also, the two terms in the integrand cancel. We then have

$$\frac{\partial^2 \mathcal{A}}{\partial t^2} = \left[ \mathcal{C}(\epsilon) \frac{\partial \epsilon}{\partial x} \right]_0^x + \frac{d^2}{dt^2} \mathcal{A}(0, t), \quad (168)$$

which in view of (162) becomes

$$\frac{\partial^2 \mathcal{A}}{\partial t^2} = \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial \epsilon}{\partial x} + \left\{ \frac{\partial^2 \mathcal{A}}{\partial t^2} - \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial \epsilon}{\partial x} \right\}_{x=0}, \quad (169)$$

or

$$\frac{\partial^2 \mathcal{A}}{\partial t^2} = \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial^2 \mathcal{A}}{\partial x^2} + \left\{ \frac{\partial^2 \mathcal{A}}{\partial t^2} - \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial^2 \mathcal{A}}{\partial x^2} \right\}_{x=0}. \quad (170)$$

We saw from (147) that the process indicated in Fig. 21 is sufficient to determine  $\epsilon$  if  $f(x)$ , the initial distribution of  $\epsilon$ , is given. From (155), however, it is evident that  $s$  is determined only to within an arbitrary function  $s(0, t)$ . If for  $x = 0$  we choose for this function one that is physically possible, then the differential equation (141) is satisfied at  $x = 0$ , the unknown left side  $(\partial^2 \mathcal{A} / \partial t^2)_{x=0}$  being determined by the known right side

$$\left\{ \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial^2 \mathcal{A}}{\partial x^2} \right\}_{x=0}, \text{ which is } \left\{ \left[ \mathcal{C}(\epsilon) \right]^2 \frac{\partial \epsilon}{\partial x} \right\}_{x=0}.$$

It follows that the large bracket in (170) vanishes, and (170) becomes

$$\frac{\partial^2 u}{\partial t^2} = [c(\epsilon)]^2 \frac{\partial^2 u}{\partial x^2},$$

which duplicates the desired differential equation (141). We have thus shown that the idea of independent propagation of infinitesimal strain increments, as described on page 62 and indicated in Fig. 21 is valid.

In Fig. 21  $f(x)$  is shown with a positive slope. This slope may, however, be negative, as shown in Fig. 22—the above mathematical treatment is valid in both cases.

From the denominators of (152) and (153) we see that if the slope of  $f(x)$  is positive, and if  $c'(\epsilon)$  is positive; then after a certain time has elapsed  $\partial \epsilon / \partial x$  and  $\partial \epsilon / \partial t$  become infinite. For strain  $\epsilon$  this time is given by

$$1 - f'c't = 0$$

or

$$t = \frac{1}{f'c'}. \quad (171)$$

Here  $f'$  is the initial value of  $\partial \epsilon / \partial x$  for the specified value of  $\epsilon$ , as is evident from (150). The smallest value of  $t$  given by (171) for our range of values of  $\epsilon$  is the time required for a singularity to develop. At this time the validity of the above mathematical treatment ceases; and at subsequent times some extension of this treatment is necessary.

## Appendix 2

Here some additional comments will be made on the problems that arise when the stress-strain curve is concave upward. These comments are mainly due to only one of the authors (HJH). A concave-upward curve is shown in Fig. 12. The smooth curve may be approximated by two or more chords, as shown, joining, for example, the points 0,  $(\sigma_1, \epsilon_1)$ , and  $(\sigma_2, \epsilon_2)$ . If only a single chord, joining the points 0 and  $(\sigma_2, \epsilon_2)$  is used, a strain wave of height  $\epsilon_2$  will travel with velocity  $\sqrt{\sigma_2/\rho\epsilon_2}$ . The equations of continuity and momentum will be satisfied, and the equation of energy will also be satisfied for the linear stress-strain curve. In this linear case it is well known that the total energy is half kinetic energy and half strain energy. But strain energy is found by integrating under the stress-strain curve from 0 to  $\epsilon_2$ . If we use the linear curve, energy is conserved but if we use the actual curve, or the two-chord approximation to it, we find that the potential energy stored in the string is below that stored in the linear case. If we add the potential and kinetic energies we find that they do not total the amount of work done in generating the wave.

A way out of this dilemma that is often adopted is to assume that a square wave of height  $\epsilon_2$  travels with a velocity given by the slope of the single chord, and to assume that the work done in generating the wave goes into the normal amount of kinetic energy, into a lesser amount of strain energy (corresponding to the area under the curve that is concave upward), and into heat. The amount of heat generated is just sufficient to conserve energy, but the way in which it is generated is usually not specified. Some insight into how the extra energy might be converted into heat is given by two problems solved in the body of the present report: the problem of the equal masses joined by slack strings, and the problem in which a disturbance consisting of a wave of fixed height was assumed, with an oscillatory wave superposed on it. In each of these problems it is possible to satisfy the equations of momentum, continuity, and energy.

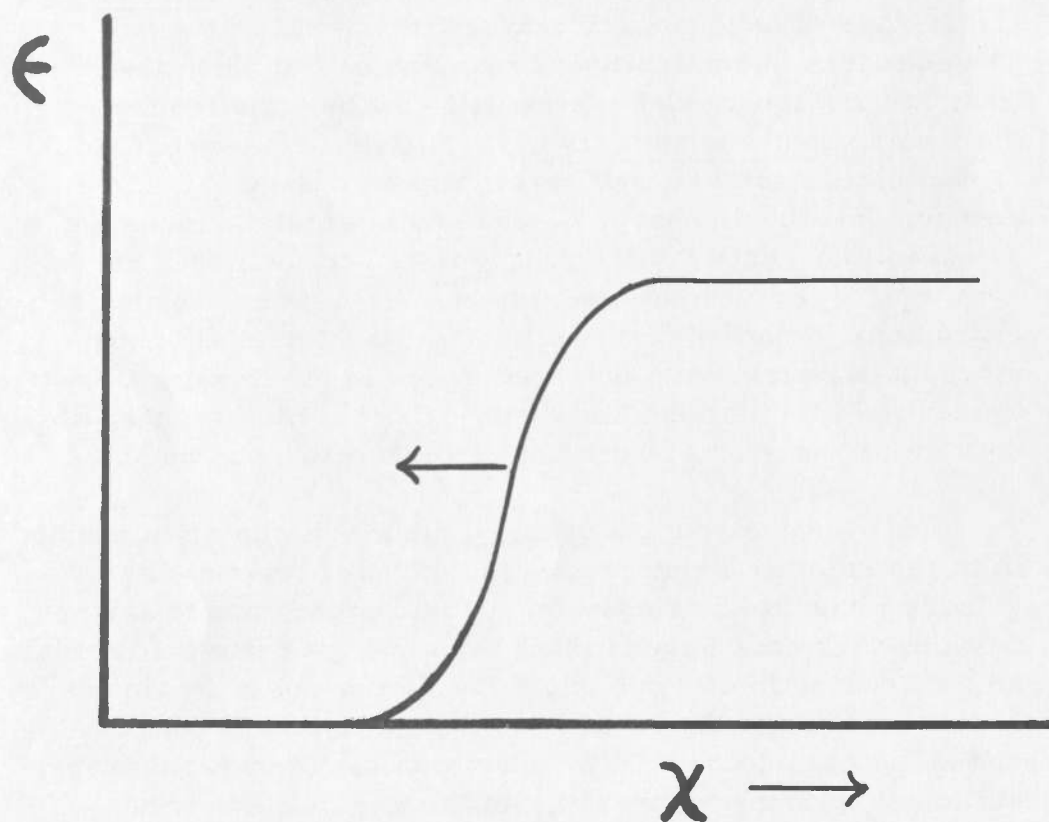


Fig. 23. Schematic diagram of the front of a square wave, with the  $x$ -dimension greatly magnified.

A possibility exists, however, that the model usually adopted is incorrect. If this is true, the treatments usually given, including a part of what has been presented in the present report, do not correspond to reality. They are self-consistent and formally correct but they may still be physically incorrect. To see where the usual assumptions may be wrong, it is necessary to consider the nature of a wave of tension as it propagates into a string. We will assume that the wave is a square wave. Mathematically the wave is infinitely steep, but physically it must certainly have a finite thickness. This thickness may be very small, but it is still several atomic spacings. Such a wave is shown schematically in Fig. 23 with the thickness greatly magnified. Now consider the speed of propagation of this wave as determined from the smooth stress-strain curve of Fig. 12.

The leading edge of the wave, where the strain is very small, must propagate with the velocity

$$c_0 = \sqrt{(\partial \sigma / \partial \epsilon)_0 / \rho} \quad (172)$$

where the subscript 0 indicates the origin in Fig. 12. Where the strain is higher, the propagation velocity is higher than  $c_0$ . Hence by the usual logic a wave that is not initially steep will become steeper because the regions of high strain travel faster than those of lower strain. However, there is no physical mechanism that will permit the steepening to go on indefinitely. The speed of propagation of a small strain into the unstrained material is  $c_0$  and the entire wave can propagate no faster.

To emphasize this point we will mention that the elastic properties of a material are determined by the forces between the various atoms and molecules that make it up. When the material is unstrained the molecules are at their equilibrium distance apart and a small strain propagates with velocity  $c_0$ . Only after the molecules have moved ~~farther~~ apart does the material have a higher value of  $\partial \sigma / \partial \epsilon$  and a higher strain velocity. In contrast, if we were dealing with a compression wave rather than

a tensile wave, the speed of the wave would not be directly determined by the intermolecular force curves. In a compression wave there is no practical limit on the magnitude of the force that can be applied to the rear surface of the shock by the fully strained material. The pressure simply builds up until the requisite amount of force is applied. Interatomic repulsive forces become very great as a material is compressed and so there is no practical limit to the process, even when the particle velocity behind the wave becomes very large. In a tensile wave, on the contrary, a force that is too large will simply pull the molecules apart. After the specimen is ruptured, no further work can be done on the unstrained material.

This situation is reminiscent of what happens in a gas, where a compression shock, traveling with a velocity greater than that of sound, is possible. Analogous expansion shocks in gases do not exist. In expansion we find only expansion waves, and no propagation velocities exceed the speed of sound. The analogy between the behavior of a strain wave and a gaseous shock wave is not exact, however, and should be used with care.

If the arguments given above are correct, no tensile strain wave can propagate into a material with a velocity greater than  $c_0$ . The model of a shock wave of height  $\epsilon_2$  traveling with velocity  $c = \sqrt{\sigma_2 / \rho \epsilon_2}$  as required by eq (51) and Fig. 12 becomes untenable because  $c > c_0$ . Since a shock wave of height  $\epsilon_2$  and velocity  $c$  is known to satisfy both continuity and momentum conditions (but not energy) do we not make matters worse by fixing the velocity at  $c_0$ ? We do not, of course, if the forward propagation velocity actually is  $c_0$ .

The problem of the merging of two finite waves, one overtaking the other, was treated in the body of this report. The stress-strain curve for that problem was two intersecting straight lines such as those shown in Fig. 12, and the wave heights were  $\epsilon_1$  and  $\epsilon_2 - \epsilon_1$  respectively, so that each pulse propagated as a square wave. When the second wave overtook the first, the propagation velocity of the merged wave abruptly changed to a value higher than that of the leading wave and lower than that of the overtaking wave. However, on the basis of the

discussion given above, we expect the propagation velocity of the merged wave to be the same as that of the leading wave, with a sudden change in strain and a sudden change in particle velocity originating at the point of overtaking. In order to satisfy the conservation equations, a reflected wave is required. This wave, since it travels in strained material, will have the higher propagation velocity of the second wave.

One of the authors (HJH) made several attempts to calculate the various waves and show how continuity, momentum, and energy conditions could all be satisfied. He did not get a satisfactory solution. Whether this was due to insufficient mathematical prowess or whether the problem was not set up in a suitable form is not known. Most of the assumed solutions that were tried out began by assuming waves of constant amplitude propagating in both directions. It may well be that waves that change in amplitude as they propagate away from the point of overtaking are required. Whatever the trouble, there seems to be no reason why a solution of the overtaking problem, with the merged wave having a propagation velocity of  $c_0$ , could not be obtained. This should lead to an understanding of what happens when the stress-strain curve is a smooth curve, concave upward. Then comparison with experiments could be made.

Some attempts were made to solve the overtaking problem involving two square waves by computer methods. A stress-strain curve consisting of two straight lines was accepted, with constants chosen so that the second wave would travel twice as fast as the first. Our thought was that we could plot the results and determine the speeds of propagation of both waves, and perhaps also those of the advancing and reflected waves formed at the point of overtaking. The attempts were unsuccessful. Our iterative method was able to deal with linear stress-strain curves and with curves that were concave downward. It was able to use a curve that was moderately concave upward if the initial strain pulse had a gradual rise, presumably because the concavity steepened the pulse without making it entirely square. But the iterative process, when used with a stress-strain curve that was concave upward, seemed always to fail when a square pulse was programmed initially and also when a strain pulse that rose gradually was sharpened by propagation.

### References

1. P. D. Crout, M. N. Pilsworth, Jr. and H. J. Hoge "Response of Polymers to Tensile Impact. 1. An integral equation, successive-substitution solution for use in digital computers, and its application to a linear elastic model", U.S. Army Natick Laboratories, Technical Report 69-18-PR (July 1968), 71 p; AD- 682714
2. T. von Karman and P. Duwez, "The propagation of plastic deformation in solids", J. Appl. Phys. 21, 987-94 (1950).
3. G. L. Taylor, Scientific Papers of G. L. Taylor (Cambridge University Press, New York, 1958), Vol. 1, Paper No. 30, 456-63, and Paper No. 32, 467-79.
4. Kh. A. Rachmatulin, Prikl. Mat. i Mech. 9, 91-100 (1945).
5. M. N. Pilsworth, Jr. and H. J. Hoge, "Rate-dependent response of polymers to tensile impact", Textile Res. J. 35, 129-39 (1965).
6. M. N. Pilsworth, Jr., "Tensile impact on rubber and nylon", Quartermaster Research and Engineering Center, Natick, Mass. Tech. Rept. PR-3 (May 1962), 66 p; AD-276525.
7. J. C. Smith and C. A. Fenstermaker, "Strain-wave propagation in strips of natural rubber subjected to high-velocity transverse impact", J. Appl. Phys. 38, 4218-24 (1967).
8. D. R. Petterson, G. M. Stewart, F. A. Odell, and R. C. Maheux, "Dynamic distribution of strain in textile materials under high-speed impact", Textile Res. J. 30, 411-31 (1960).
9. P. D. Crout, Research Notebooks on Strain-Wave Propagation, No. 1, begun Sep. 1961, 152 p; No. 2, begun Sep. 1963, 152 p; and No. 3, begun Feb. 1966, 56 p. A set of these notebooks (Xerox copies of the originals) has been filed as an appendix to copy 1 of Reference 1 in the Technical Library of the U. S. Army Natick Laboratories.

## DISTRIBUTION

Mr. Anthony L. Alesi  
Army Materials and Mechanics  
Research Center  
Watertown, Mass. 02172

Prof. Stanley Backer  
Textile Division  
Dept. of Mechanical Engineering  
Massachusetts Institute of Technology  
Cambridge, Mass. 02139

Mr. F. G. Boccetti  
Dacron Research Laboratory Library  
E. I. Dupont Co.  
Kinston, North Carolina 28501

Prof. P. C. Chou  
Drexel Institute of Technology  
Philadelphia, Pa. 19104

Mr. R. E. Connell  
Central Research Laboratory  
Allied Chemical Corp.  
Box 309  
Morristown, N.J. 07960

Mr. Mark L. Dannis  
Senior Research Associate  
B. F. Goodrich Research Center  
Brecksville, Ohio 44141

Mr. L. R. Easley  
Bldg. 59  
Tennessee Eastman Company  
P. O. Box 511  
Kingsport, Tenn. 37662

Mr. W. Julian Ferguson  
Naval Research Laboratory  
Washington, D. C. 20390

Prof. Francis B. Hildebrand  
Mathematics Department  
Massachusetts Institute of Technology  
Cambridge, Mass. 02139

Dr. John D. Hoffman  
Organic and Fibrous Materials  
National Bureau of Standards  
Washington, D. C. 20234

Prof. H. Kolsky  
Brown University  
Providence, Rhode Island 02912

Dr. A. M. Kotliar  
Central Research Laboratory  
Allied Chemical Corporation  
Box 309  
Morristown, N. J. 07960

Dr. Sudhir Kumar  
U. S. Army Research Office, Durham  
Box CM, Duke Station  
Durham, North Carolina 27706

Prof. E. H. Lee  
Applied Mechanics Department  
Stanford University  
Stanford, California 94305

Dr. W. James Lyons  
Textile Research Institute  
P. O. Box 625  
Princeton, New Jersey 08540

Dr. Henry M. Morgan  
Old Concord Road  
Lincoln, Mass. 01773

DISTRIBUTION (cont'd)

Mr. Broham Norwick  
Beaunit Textiles  
450 Seventh Avenue  
New York, N.Y. 10001

Plastics Technical Evaluation  
Center  
Attn: Harry E. Peibly, Jr., Chief  
Picatinny Arsenal  
Dover, New Jersey 07801

Mr. Jason Prumel  
Central Research Laboratory  
Allied Chemical Corp.  
Box 309  
Morristown, New Jersey 07960

Dr. Jack C. Smith  
Institute for Materials Research  
National Bureau of Standards  
Washington, D. C. 20234

Mr. George M. Stewart  
Biophysics Division  
Army Chemical Center, Maryland 21401

Miss Judith Taylor  
Rohm and Haas Company  
Library, Box 219  
Bristol, Pa. 19007

Dr. Guy Waddington, Director  
Office of Critical Tables  
National Academy of Sciences  
2101 Constitution Avenue  
Washington, D. C. 20418

Professor George P. Wadsworth  
Mathematics Department  
Massachusetts Institute of  
Technology  
Cambridge, Mass. 02139

Mr. Charles Warburton  
Textile Research Institute  
Princeton, New Jersey 08540

Dr. M. T. Watson  
Bldg. 150  
Tennessee Eastman Co.  
P. O. Box 511  
Kingsport, Tennessee 37662

Mr. Louis C. Weiss  
Fiber Physics Investigations  
U. S. Dept. of Agriculture  
P. O. Box 19687  
New Orleans, La. 70119

Dr. Anthony F. Wilde  
Army Materials and Mechanics  
Research Center  
Watertown, Mass. 02172

Dr. Lawrence A. Wood  
Institute for Materials Research  
National Bureau of Standards  
Washington, D. C. 20234

UNCLASSIFIED

Security Classification

## DOCUMENT CONTROL DATA - R &amp; D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION	
U.S. Army Natick Laboratories Natick, Massachusetts 01760		Unclassified	
		2b. GROUP	
3. REPORT TITLE Response of Polymers to Tensile Impact. 2. Extension of the Integral-Equation, Successive-Substitution Solution to Non-Linear, Time-Independent Materials			
4. DESCRIPTIVE NOTES (Type of report and Inclusive dates)			
5. AUTHOR(S) (First name, middle initial, last name) Prescott D. Crout and Malcolm N. Pilsworth, Jr., Harold J. Hoge			
6. REPORT DATE February 1974	7a. TOTAL NO. OF PAGES 76	7b. NO. OF REFS 9	
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)		
b. PROJECT NO. 1T061102B11A-07	74-26-PR		
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
d.			
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY U.S. Army Natick Laboratories Natick, Massachusetts 01760	
13. ABSTRACT An integral-equation, successive-substitution solution for the propagation of strain waves in a linear elastic material, developed in an earlier paper, is here extended to non-linear materials without creep (time-independent materials). A previously-published problem, based on experimental data for nylon string is solved by the successive-substitution method and compared with the earlier solution obtained by the method of characteristics. The agreement of the two solutions is at first good, but in the region where the stress-strain curve of the nylon is concave upward a tendency to oscillation builds up and the solution eventually oscillates. This occurs in approximately the same region where shock waves can appear in a method-of-characteristics solution. Oscillations in the strained portion of a string are examined theoretically and it is shown that the energy of standing waves can be used to obtain energy conservation. Although the successive-substitution method generates oscillations when it has to deal with a discontinuity in strain, the possibility that oscillations also actually exist as a means of conserving energy cannot be ruled out.			

DD FORM 1473  
1 NOV 65REPLACES DD FORM 1473, 1 JAN 64, WHICH IS  
OBSOLETE FOR ARMY USE.

Security Classification

**Security Classification**

14.

### KEY WORDS

LINK A

LINK B

LINK C

ROLE

WT

ROLE

WT

ROLE

WT

8.9

8,9

8,9

8,9

8,9

9

9

9

9

10

10

10

10

10

**Security Classification**